INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

Non-stationary response statistics of nonlinear oscillators with fractional derivative elements under evolutionary stochastic excitation

Introduction

A novel approximate technique for determining the non-stationary response amplitude probability density function (PDF) of nonlinear / hysteretic oscillators endowed with fractional derivative elements and subjected to evolutionary stochastic excitation is developed. Specifically, resorting to a stochastic linearization / averaging treatment of the problem yields a first-order stochastic differential equation governing the oscillator response amplitude. Next, assuming a time-dependent PDF of the Rayleigh kind for the response amplitude, the associated Fokker-Planck partial differential equation is solved for determining the oscillator non-stationary response amplitude PDF in closed-form and at a minimal computational cost. An additional advantage of the technique is that it can handle arbitrary forms of the excitation evolutionary power spectrum, even of the non-separable kind.

Mathematical formulation

A nonlinear oscillator with fractional derivative terms is considered, whose governing equation of motion is given by

$$\ddot{x}(t) + \beta \mathcal{D}_{0,t}^{\alpha} x(t) + z(t, x, \dot{x}) = w(t)$$

where $z(t, x, \dot{x})$ represents an arbitrary nonlinear function that can also account for hysteretic behaviors; and $w(t)$ denotes a Gaussian, zero-mean, non-stationary stochastic process with an evolutionary broad-band power spectrum $S(\omega, t)$. Further, β is a coefficient and $\mathcal{D}_{0,t}^{\alpha} x(t)$ denotes a Caputo fractional derivative defined as

$$\mathcal{D}_{0,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} d\tau$$

for $0 < \alpha < 1$. Resorting to the assumption of light damping for the oscillator, it can be argued that it exhibits a pseudo-harmonic behavior, where the oscillator response amplitude $A(t)$ and phase $\psi(t)$ are considered to be slowly varying functions with respect to time, and approximately constant over one cycle of oscillation. Application of the statistical linearization methodology results to the equivalent linear oscillator

$$\ddot{x}(t) + [\beta + \beta(A)]\dot{x}(t) + \omega^2(A)x(t) = w(t)$$

and an error minimization procedure in the mean square sense yields the equivalent linear amplitude-dependent damping and stiffness coefficients $\beta(A)$ and $\omega^2(A)$, respectively.

A stochastic averaging technique can be applied to the linearized system with the aim of reducing its order, and potentially its complexity from a solution perspective. This yields a first-order stochastic differential equation for the response amplitude $A(t)$. The corresponding Fokker-Planck partial differential equation governing the evolution in time of the response amplitude PDF is given by

$$\frac{\partial p(A,t)}{\partial t} = -\left\{\left[\frac{1}{2}\beta_0 + \beta(A)\right] A - \frac{\beta_0 \omega_0^2}{2A\omega^2(A)}\right\} \frac{\partial p(A,t)}{\partial A} + \left[\frac{\beta_0 \omega_0^2}{2\omega^2(A)}\right] \frac{\partial^2 p(A,t)}{\partial A^2}$$

Motivated by the solution to the Fokker-Planck equation in the stationary case, a novel approximate analytical solution is developed for the non-stationary response amplitude PDF $p(A,t)$ of the general nonlinear oscillator. This takes the form

$$p(A,t) = \frac{\sin\left(\frac{\alpha\pi}{2}\right) A}{\omega_0^{1-\alpha} c(t)} exp\left(-\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}} \frac{A^2}{2c(t)}\right)$$

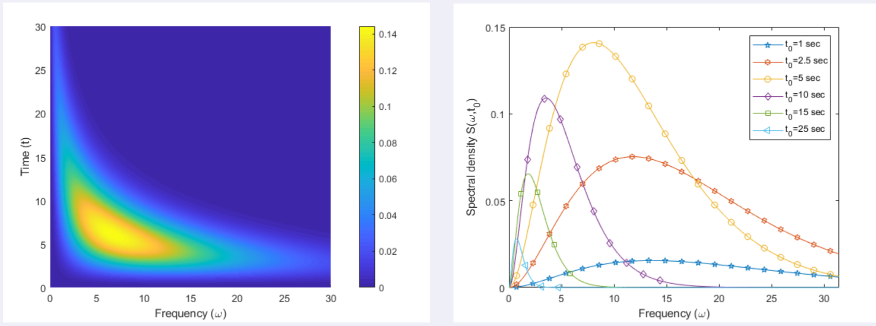
where $c(t)$ is a time-dependent coefficient to be determined. Evaluating the second moment of the oscillator response yields the oscillator non-stationary response variance

$$E(x^2) = \frac{\omega_0^{1-\alpha}}{\sin\left(\frac{\alpha\pi}{2}\right)} c(t)$$

Numerical example

The bilinear hysteretic nonlinear oscillator with fractional derivative elements is considered for assessing the reliability of the developed technique. The initially at rest oscillator, is subjected to non-stationary stochastic excitation described by the non-separable evolutionary power spectrum

$$S(\omega,t) = S_0 \left(\frac{\omega}{5\pi}\right)^2 \exp(-c_0 t)\, t^2 \exp\left(-\left(\frac{\omega}{5\pi}\right)^2 t\right)$$



$S(\omega, t)$ comprises some of the main characteristics of seismic shaking, such as decreasing of the dominant frequency with time.

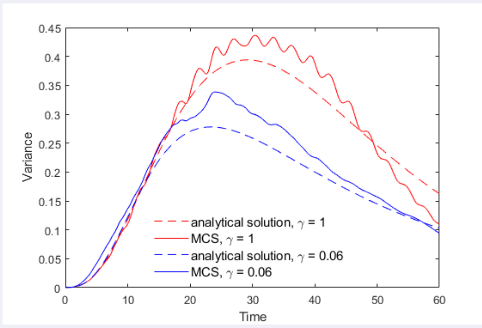
Denoting by $x^{\star}$ the critical value of the displacement at which the yield occurs, and by ω_0 the oscillator natural frequency corresponding to the primary elastic slope, the non-dimensional displacement $y = \frac{x}{x^{\star}}$ and time quantity $\tau = \omega_0 t$ are employed. The restoring force of the oscillator becomes

$$z(t, y, \dot{y}) = \gamma y + (1-\gamma) z_0$$

where γ denotes the post- to pre-yield stiffness ratio, and z_0 is the hysteretic force corresponding to the elasto-plastic characteristic, described by the first order differential equation

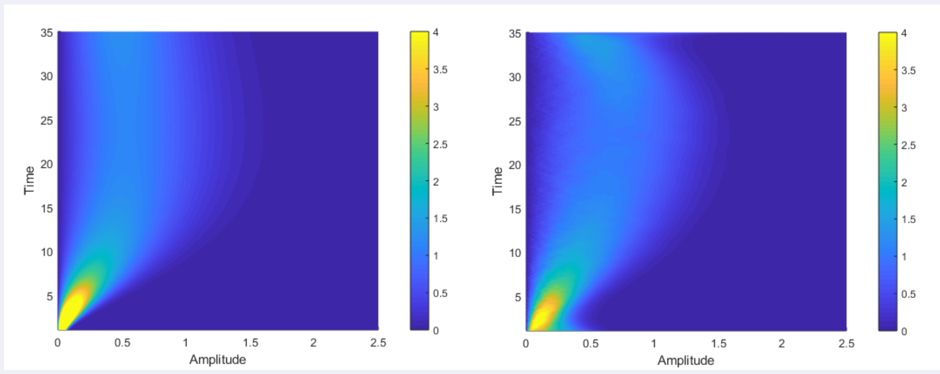
$$\dot{z_0} = \dot{y}\big(1 - H(\dot{y})H(z_0 - 1) - H(-\dot{y})H(-z_0 - 1)\big)$$

The excitation and system parameter values used are: $S_0 = 0.08$, $c_0 = 0.12$, $\omega_0 = 2.34$, $\beta = 0.1$, $\gamma = 0.06$.

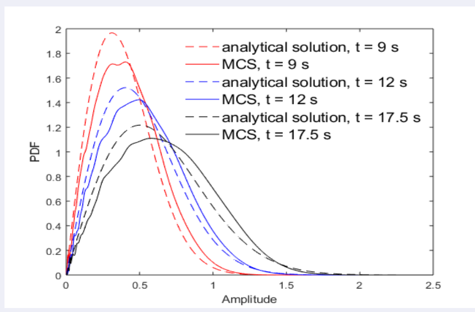


Non-stationary response variance of a bilinear hysteretic oscillator ($\gamma = 0.06$) with fractional derivative order $\alpha = 0.5$

The closed-form expression for determining the non-stationary response amplitude PDF $p(A,t)$ is also assessed.



Non-stationary response amplitude PDF of a bilinear hysteretic oscillator ($\gamma = 0.06$) with fractional derivative order $\alpha = 0.5$: analytical PDF (left); MCS-based estimate (10,000 realizations)(right)



Analytical vis-à-vis MCS-based (10,000 realizations) response amplitude PDFs of a bilinear hysteretic oscillator ($\gamma = 0.06$) with fractional derivative order $\alpha = 0.5$, plotted for various time instants

Reference

[1] Fragkoulis, V.C.; Kougioumtzoglou, I.A.; Pantelous, A.A.; Beer, M. (2019). Nonlinear Dynamics. doi.org/10.1007/s11071-019-05124-0