INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

Breaking the double loop

Operator norm theory as an efficient tool to calculate imprecise probabilities

Introduction

Let $\mathcal{M}$ represent a function that maps a set of n_x input parameters $\boldsymbol{x} \in D_x \subseteq \mathbb{R}^{n_x}$, with D_x a set of feasible input parameters to a set of output parameters $y \in \mathbb{R}^{n_y}$ via following relationship:

$$\boldsymbol{y} = \mathcal{M}(\boldsymbol{x})$$

where $\mathcal{M}$ may represent a numerical. In order to express aleatory uncertainty in the model parameters, they are usually modelled as random variables$X = (X_1, X_2, \dots X_{n_x})$, the distribution of which is described by the probability density function $f_{\boldsymbol{X}}$. Given this description, an analyst is usually interested in approximating the probability of failure $p_f = P(\mathcal{M}(\boldsymbol{x}) \leq 0)$, where $\mathcal{M}(\boldsymbol{x})$ represents the performance function that indicates whether the design failed ($\mathcal{M}(\boldsymbol{x}) \leq 0$) or not:

$$p_f = \int_{D_X} I_{\mathcal{M}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) \mathrm{d}x$$

with $I_{\mathcal{M}}(\boldsymbol{x})$ a function that has value 1 in case $\mathcal{M}(\boldsymbol{x}) \leq 0$ and 0 otherwise.

In most real-life applications, an analyst has only partial information about $f_{\boldsymbol{X}}(\boldsymbol{x})$ due to often imprecise, diffuse, fluctuating, incomplete or vague nature of the available data. In this case, parametric p-boxes might provide the analyst with a viable tool to assess the sensitivity of p_f to this lack-of-knowledge. These parametric p-boxes are described by a family of CDFs whose parameters $\theta_i \in \mathbb{R}$ are unknown up to the property that they must be contained within intervals $\theta^I = [\underline{\theta_i}, \overline{\theta_i}], i = 1, \dots, n_\theta$. Also the model $\mathcal{M}$ can be described by parameters $\vartheta_j \in \mathbb{R}, j = 1, \dots, n_\vartheta$ bounded by an interval $\vartheta_j^I = [\underline{\vartheta}_j, \overline{\vartheta}_j]$. As such, also p_f becomes an explicit function of $[\theta, \vartheta]$. The lower bound on p_f is obtained as:

$$\underline{p_f} = \min_{\theta \in \theta^I, \vartheta \in \vartheta^I} p_f(\theta, \vartheta) = \min_{\theta \in \theta^I, \vartheta \in \vartheta^I} \int_{D_X} I_{\mathcal{M}}(\boldsymbol{x}, \boldsymbol{\vartheta}) f_{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{\theta}) \mathrm{d}x$$

and the upper bound can be obtained similarly. However, in practice, the solution of these two double-loop problems is computationally intractable when realistic numerical models are applied due to the required repeated solution of the reliability problem in the inner loop. This study therefore presents an efficient approach to effectively decouple these loops.

The operator norm framework to decouple the double loop

The operator norm framework, as introduced in [1] allows to recast the double loop calculations corresponding to finding the bounds on p_f into two deterministic optimization problems and two reliability calculations, breaking this double loop. Hereto, first the epistemic parameters that yield $\underline{p_f}$ and $\overline{p}_f$, resp. $[\theta^L, \vartheta^L]$ and $[\theta^U, \vartheta^U]$, are determined by optimizing over the operator norm of the linear map that represents $\mathcal{M}$:

$$[\theta^L, \vartheta^L] = \mathrm{armin}_{\theta \in \theta^I, \vartheta \in \vartheta^I} \|A(\theta, \vartheta)\|_{\infty,2}$$

$$[\theta^U, \vartheta^U] = \mathrm{armax}_{\theta \in \theta^I, \vartheta \in \vartheta^I} \|A(\theta, \vartheta)\|_{\infty,2}$$

where$\|\cdot\|_{\infty,2}$ represents the $2/\infty$ operator norm and $A(\theta, \vartheta)$ represents the linear map that represents the model $\mathcal{M}$ according to following relationship:

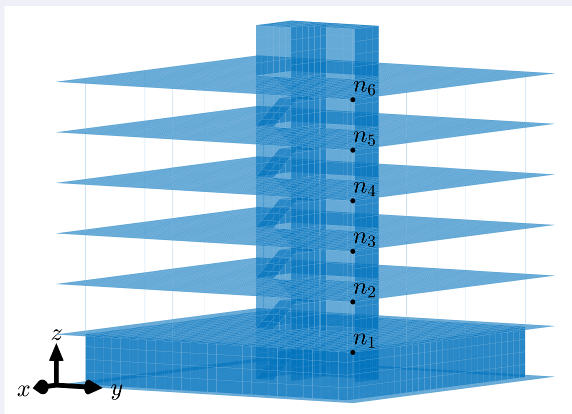
$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{\vartheta}) x(\boldsymbol{\theta}) = \boldsymbol{A}(\boldsymbol{\vartheta}) \boldsymbol{B}(\boldsymbol{\theta}) \boldsymbol{\xi}$$

where the matrix $\boldsymbol{A}$ represents the model and $\boldsymbol{B}$ represents an appropriate transformation of the standard Normal random vector ξ. Then $[\theta^L, \vartheta^L]$ and $[\theta^U, \vartheta^U]$ can be used to compute the corresponding failure probabilities. For details and mathematical derivations, the interested reader is referred to [1,2].

Example: a six-story building

The method is applied to a six-story reinforced concrete building shown below. Each floor is a square with length of 32m and all floor slabs are 20cm thick and are supported by a C-shaped shear wall and 16 square columns. The building is subjected to a Gaussian stochastic process load with modulated Clough-Penzien spectrum $S^{CP}(\boldsymbol{\theta})$.The 7 filter and modulation parameters of this spectrum are considered to be interval valued. Furthermore, Young's modulus of the concrete of the 6 floors is also interval-valued. As such, 13 independent interval parameters are used to describe the epistemic uncertainty on top of the stochastic load.

Failure is defined as the first passage of one of the inter-story drifts exceeding 2mm. The reliability problem for one realisation of the epistemic uncertain parameters is computed using Directional Importance Sampling with 500 samples, whereas the outer loop is solved using vertex method or quasi-Monte Carlo simulation (double loop Monte Carlo). These results are compared to those obtained via the introduced operator norm framework.



The table below clearly illustrates that the operator norm framework can predict the bounds on p_f at far reduced computational cost, as evidenced by the number of required model evaluations. In fact, most of the model evaluations for the operator norm method are required during the deterministic optimization to determine $[\theta^L, \vartheta^L]$ and $[\theta^U, \vartheta^U]$. Furthermore, the bounds obtained via the operator norm are more generally applicable since no assumptions regarding monotonicity must be made as is the case with the Vertex analysis [1,2]. Further work will explore the application of statistical linearization to expand the method to bound p_f in combination with nonlinear numerical models.

		$\|A(\theta,\vartheta)\|_{\infty,2}$	p_f	# modeleval
Vertex analysis	L	0.0012	6.23×10^{-08}	4096000
	U	0.0025	0.0855	
Operator norm	L	0.0012	6.59×10^{-08}	3500
	U	0.0025	0.0894	2600
Double loop Monte Carlo	L	0.0013	9.04×10^{-07}	5000000
	U	0.0023	0.0481	

dr.-Ing. M. Faes
Prof.dr. M. Valdebenito
Prof.dr.-Ing. M. Beer

Institute for Risk and Reliability
www.irz.uni-hannover.de

References

[1] Faes, M., Valdebenito, M., Moens, D., & Beer, M. (2020). *Computers & Structures*, *239*, 106320.

[2] Faes, M, Valdebenito, M. , Moens, D., & Beer, M. (2020). Operator norm theory as an efficient tool to propagate hybrid uncertainties and calculate imprecise probabilities. *Preprint Submitted to MSSP*.

fwo Research Foundation Flanders Opening new horizons

Alexander von Humboldt Stiftung / Foundation