Harmonic wavelets based response evolutionary power spectrum determination of linear and nonlinear structural systems with singular matrices

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Abstract

A new approximate analytical technique is proposed for determining the response evolutionary power spectrum (EPS) of stochastically excited structural multi-degree-of-freedom (MDOF) linear and nonlinear systems with singular matrices. Such systems can appear, indicatively, when a redundant coordinates modeling is adopted for forming the equations of motion of complex multi-body systems. For this case, it can be argued that this modeling approach facilitates the system's stochastic response analysis, since employment of redundant DOFs is associated with computational cost efficient solution frameworks, and potentially provides with enhanced modeling flexibility. In this context, aiming at the joint time-frequency response analysis of MDOF systems, recently developed wavelet-based solution frameworks, which generalize classic input-output relationships of random vibration, are adopted and further generalized in this paper to account for systems with singular matrices. Specifically, resorting to the theory of generalized inverses of singular matrices, as well as to the theory of harmonic wavelets, a Moore-Penrose generalized matrix inverse excitation-response relationship is derived herein for determining the response EPS of linear MDOF systems. Further, a recently developed harmonic-wavelet-based statistical linearization technique is also generalized to account for the case of nonlinear MDOF systems. The validity of the proposed technique is demonstrated by pertinent numerical examples.

Keywords: Stochastic Dynamics - Moore-Penrose Inverse - Harmonic Wavelet - Singular Matrix - Evolutionary Power Spectrum - Time-Frequency Analysis

1 Introduction

The nature of environmental excitations, such as earthquakes and wind loadings which evolve in time and are described by evolutionary power spectra (EPS) (1), plays an instrumental role in the efficient analysis of structural systems. In conjunction with the complexity of the considered system, they constitute two critical aspects towards the development of efficient response analysis solution treatments. Therefore, taking into account the non-stationary characteristics of natural excitations, several frameworks have been proposed for determining the system response, as well as for conducting joint time-frequency response analysis, of linear and nonlinear systems; see, indicatively, Refs. (2; 3; 4; 5; 6; 7; 8; 9; 10; 11).

As far as the joint time-frequency response analysis of engineering systems is concerned, the advent of the potent machinery of wavelets has been proved pivotal. It has substantially enhanced the arsenal of the system response characterization methods, when dynamic systems subjected to non-stationary excitation are considered

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(12; 8). Specifically, standard excitation-response relationships of random vibration theory, have been extended and generalized in a wavelet-based framework, whereas wavelet-based versions of classic methodologies and techniques, such as the statistical linearization method (2), have also been developed (9; 13). Moreover, wavelet analysis has been used in conjunction with fractional calculus, for deriving the EPS of oscillators endowed with fractional derivative elements (14).

With regards to the complexity of the dynamic systems, it is commonly accepted that the minimum number of independent coordinates/degrees-of-freedom (DOF) is used in forming the system governing equations of motion (2). However, it can be argued that for the case of multibody systems, a redundant coordinates approach potentially facilitates the modeling procedure, while also results in enhanced flexibility and reduced overall computational cost (15; 16; 17; 18; 19; 20; 21; 22). Further, it also leads to systems with singular matrices, rendering all standard stochastic response analysis frameworks inapplicable. Note, in passing, that singular matrices do not solely appear due to adopting a redundant DOFs modeling. They are also met in diverse engineering systems and applications, such as systems with "massless" joints (23; 24), and in vibratory energy harvesting applications, where they appear due to the coupling between the governing equations of the mechanical and the electrical system (25).

In this paper, taking into account the aforementioned aspects, a generalized inverse matrix harmonic-wavelet-based treatment is proposed for conducting joint time-frequency response analysis of linear and nonlinear MDOF systems with singular matrices, which are subjected to non-stationary excitation. Specifically, focusing on the determination of the system response EPS, and resorting to the theory of the Moore-Penrose (M-P) generalized matrix inverses, standard harmonic-wavelet-based techniques (9; 13) are generalized herein.

In this context, adopting a redundant coordinates modeling of the equations of motion, and also employing the locally stationary wavelet (LSW) representation of a stochastic process (9), an M-P localized in time and frequency domains, harmonic-wavelet-based frequency response function (MP HW-FRF) is derived. This can be construed as a generalization of a recently developed HW-FRF (13), to account for systems with singular matrices. Further, the MP HW-FRF is used for determining the system response EPS, by constructing an input-output relationship which connects the excitation and response EPS. Next, a recently developed harmonic-wavelet-based statistical linearization methodology (13) is also generalized. In this regard, an equivalent linear system corresponding to the original nonlinear system is defined, and a set of time and frequency dependent expressions for the equivalent linear elements is derived. This is attained by resorting to an iterative solution numerical scheme. The scheme is applied on the coupled set of equations defined by the set of the equivalent elements expressions, and the M-P input-output relationship of the equivalent linear system. Further, although the employment of the M-P inverse framework implies a family of equivalent linear elements, uniquely defined elements are determined by setting equal to zero the arbitrary term of the M-P based family. Finally, the nonlinear system response EPS is estimated by considering the corresponding response EPS of the equivalent linear system. The efficiency of the proposed M-P inverse framework is demonstrated by pertinent examples of linear and nonlinear MDOF systems. The obtained results are compared with results derived by the standard solution treatment of Ref. (13) and are in complete agreement.

2 Mathematical preliminaries

2.1 Aspects of Moore-Penrose matrix inverse theory

The study of generalized matrix inverses has initiated and flourished mainly as a result of attempting to solve systems of algebraic equations of the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{A}$ is either rectangular $m_0 \times n_0$, or square but singular $n_0 \times n_0$ matrix, and $\mathbf{x}$, $\mathbf{b}$ are n_0 vectors. Eq. (1) appears in many theoretical problems in mathematics as well as many applied problems. Clearly, the nature of matrix $\mathbf{A}$ renders its solution impossible. In this regard, the necessity of defining any form of "partial inverse" for rectangular or square but singular matrices gave birth to the theory of generalized matrix inverses (26). The Moore-Penrose (M-P) generalized matrix inverse holds an exceptional place among these theoretical results.

Definition 1. *For any matrix $\mathbf{A} \in \mathbb{C}^{m_0 \times n_0}$, there is a unique matrix $\mathbf{A}^+ \in \mathbb{C}^{n_0 \times m_0}$ such that*

$$\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A},\ \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+, (\mathbf{A}\mathbf{A}^+)^* = \mathbf{A}\mathbf{A}^+,\ (\mathbf{A}^+\mathbf{A})^* = \mathbf{A}^+\mathbf{A}. \tag{2}$$

The matrix $\mathbf{A}^+$ defined in Eq. (2) is called the M-P inverse of $\mathbf{A}$. If $\mathbf{A} \in \mathbb{R}^{n_0 \times n_0}$ is non-singular, then $\mathbf{A}^+$ coincides with $\mathbf{A}^{-1}$. Using the M-P inverse, a closed form solution to the algebraic system of Eq. (1) is attained, which highlights its importance for several applications. In this regard, for any matrix $\mathbf{A} \in \mathbb{R}^{m_0 \times n_0}$, Eq. (1) implies

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{I}_n - \mathbf{A}^+\mathbf{A})\mathbf{y}, \tag{3}$$

where $\mathbf{y}$ denotes an arbitrary n_0 vector and $\mathbf{I}_{n_0}$ represents the identity $n_0 \times n_0$ matrix.

By resorting to the M-P matrix inverse theory, classic solution treatments of random vibration have been recently generalized for determining the stationary stochastic response of linear and nonlinear systems with singular matrices (27; 28; 29); see also Refs. (23; 30; 31; 24) for additional applications. However, it can be argued that the application of state-of-the-art M-P solution frameworks is considerably limited, since relevant approaches completely ignore the non-stationary characteristics of the system excitation. A more detailed presentation of the M-P inverse theory can be found in Refs. (26) and (32).

2.2 Harmonic wavelets theory elements

2.2.1 Generalized harmonic wavelets

In this section, a concise presentation of the basic elements of wavelets analysis is provided for completeness. In this regard, the wavelet transform $[W_\psi f](j,k)$ of a function $f(t)$ is defined as

$$[W_\psi f](j,k) = \frac{1}{|\, j\,|^{1/2}} \int_{-\infty}^{\infty} f(t)\psi^* \left(\frac{t-k}{j}\right) \mathrm{d}t, \tag{4}$$

where $\psi(t)$ is the mother wavelet, i.e., the generating function for all basis functions, $[W_\psi f](j,k)$ is the wavelet coefficient at frequency and time scale j and k, respectively, and "*" denotes the complex conjugate operator. Eq. (4) is used for conducting joint time-frequency analysis of the function $f(t)$. Further, choosing a different mother wavelet function $\psi(t)$ results in the definition of different families of wavelets. Among them, the so-called generalized harmonic wavelets (GHW) constitute the most often considered family of wavelets in engineering applications (33; 34). Utilizing the set of parameters (m,n) and k for defining the bandwidth at all scale levels, the members of the GHW family in frequency domain are given by

$$\Psi^G_{(m,n),k}(\omega) = \begin{cases} \frac{1}{(n-m)\Delta\omega} \exp\left(-i\omega \frac{kT_0}{n-m}\right), & m\Delta\omega \leq \omega < n\Delta\omega \\ 0, & otherwise \end{cases} \tag{5}$$

where $m,n,k \in \mathbb{Z}^+$, T_0 denotes the total time duration and $\Delta\omega = 2\pi/T_0$. The importance of the GHW of Eq. (5) stems from the fact that a decoupling of the time-frequency resolution from the values of the central frequency

$$\omega_{c,(m,n),k} = \frac{(n+m)}{2}\Delta\omega, \tag{6}$$

which is defined in the intervals $[m\Delta\omega, n\Delta\omega]$ and $\left[\frac{kT_0}{n-m}, \frac{(k+1)T_0}{n-m}\right]$, is attained.

Further, the continuous generalized harmonic wavelet transform (GHWT) of a function $f(t)$ is defined as the projection of $f(t)$ on the orthogonal basis given by the family of GHWs of Eq. (5) (34), that is

$$W^G_{(m,n),k}[f] = \frac{n-m}{kT_0} \int_{-\infty}^{\infty} f(t)\overline{\psi^G_{(m,n),k}(t)} dt. \tag{7}$$

A detailed presentation of the topic is found in Refs. (7; 12; 8).

2.2.2 Locally stationary wavelet representation of non-stationary stochastic processes

In terms of engineering applications, the versatile locally stationary wavelet (LSW) representation of stochastic processes, firstly introduced in Ref. (35), facilitates the ensuing analysis by allowing for a representation of a given non-stationary process as the summation of sub-processes defined at different scales and translation levels. Specifically, adopting a GHW expansion of the system response and excitation, the LSW forms a set of orthogonal basis functions on any given finite interval. Therefore, it results in the definition of a wavelet spectrum at a particular scale and location providing, in essence, with the joint time-frequency content of the system's non-stationary excitation and the corresponding response.

In this regard, considering the family of GHWs of Eq. (7), the generalized harmonic-wavelet-based representation of an n_0 vector process $\mathbf{x}(t)$ takes the form

$$\mathbf{x}(t) = \sum_{(m,n)} \sum_{k} \mathbf{x}_{(m,n),k}(t), \tag{8}$$

where the localized process $\mathbf{x}_{(m,n),k}(t)$ at scale (m,n) and translation k, is given by

$$\mathbf{x}_{(m,n),k}(t) = \mathbf{a}_{(m,n),k}\cos\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right] + \mathbf{b}_{(m,n),k}\sin\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right]. \tag{9}$$

In Eq. (9), $\omega_{c,(m,n),k}$ denotes the central frequency of Eq. (6). Further, $\mathbf{a}_{(m,n),k}$ and $\mathbf{b}_{(m,n),k}$ are statistically independent, zero-mean vector processes, whose variance is related to the EPS matrix $\mathbf{S}^{\mathbf{xx}}_{(m,n),k}$ via the expression (9)

$$\mathbb{E}\left[\mathbf{a}_{(m,n),k}\mathbf{a}^{\mathrm{T}}_{(m,n),k}\right] = \mathbb{E}\left[\mathbf{b}_{(m,n),k}\mathbf{b}^{\mathrm{T}}_{(m,n),k}\right] = 2(n-m)\Delta\omega\mathbf{S}^{\mathbf{xx}}_{(m,n),k}. \tag{10}$$

Finally, resorting to the orthogonality properties of monochromatic functions, i.e.,

$$\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} \cos\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right]\cos\left[\omega_{c,(i,j),l}\left(t - \frac{lT_0}{j-i}\right)\right]dt = \begin{cases} \frac{T_0}{2(n-m)} & \text{, if } (m,n)=(i,j), k=l \\ 0 & \text{, otherwise} \end{cases} \tag{11}$$

and

$$\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} \sin\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right]\sin\left[\omega_{c,(i,j),l}\left(t - \frac{lT_0}{j-i}\right)\right]dt = \begin{cases} \frac{T_0}{2(n-m)} & \text{, if } (m,n)=(i,j), k=l \\ 0 & \text{, otherwise} \end{cases} \tag{12}$$

and utilizing Eqs. (9) and (10), the important for the ensuing analysis relationships

$$\mathbb{E}\left[\mathbf{x}_{(m,n),k}(t)\right] = \mathbb{E}\left[\dot{\mathbf{x}}_{(m,n),k}(t)\right] = \mathbb{E}\left[\ddot{\mathbf{x}}_{(m,n),k}(t)\right] = \mathbf{0}, \tag{13}$$

$$\mathbb{E}\left[\mathbf{x}_{(m,n),k}(t)\dot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}(t)\right] = \mathbb{E}\left[\dot{\mathbf{x}}_{(m,n),k}(t)\ddot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}(t)\right] = \mathbf{0}, \tag{14}$$

$$\mathbb{E}\left[\mathbf{x}_{(m,n),k}(t)\mathbf{x}^{\mathrm{T}}_{(m,n),k}(t)\right] = 2\mathbf{S}^{\mathbf{xx}}_{(m,n),k}(n-m)\Delta\omega, \tag{15}$$

$$\mathbb{E}\left[\dot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}(t)\dot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}(t)\right] = 2\omega^2_{c,(m,n),k}\mathbf{S}^{\mathbf{xx}}_{(m,n),k}(n-m)\Delta\omega, \tag{16}$$

$$\mathbb{E}\left[\mathbf{x}_{(m,n),k}(t)\ddot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}(t)\right] = -2\omega^2_{c,(m,n),k}\mathbf{S}^{\mathbf{xx}}_{(m,n),k}(n-m)\Delta\omega, \tag{17}$$

and

$$\mathbb{E}\left[\ddot{\mathbf{x}}_{(m,n),k}(t)\ddot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}(t)\right] = 2\omega^4_{c,(m,n),k}\mathbf{S}^{\mathbf{xx}}_{(m,n),k}(n-m)\Delta\omega, \tag{18}$$

are derived; see also Refs. (9; 36; 13).

Note, in passing, that the advantages of employing the LSW representation of Eqs. (8) and (9), are the simplicity and straightforward application of the representation model. Taking also into account its efficiency in estimating the response EPS (9; 13), it is adopted in the ensuing analysis, where the standard harmonic-wavelet-based framework of Ref. (13) is generalized to account for systems with singular matrices. However, it is also noted that a less approximate stochastic process representation than that of Eq. (9) has been recently used for joint time-frequency response analysis (37; 38). This consists in the employment of a periodized generalized harmonic wavelets (PGHWs) based framework, and is identified as a potential future extension of the herein developed M-P generalized matrix inverse theoretical framework.

3 Stochastic response of systems with singular matrices subjected to non-stationary excitation

3.1 Linear systems with singular matrices

The general form of the equations of motion of a lumped-parameter n_0-DOF linear system is given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}(t), \tag{19}$$

where $\mathbf{q}$ denotes the n_0 vector of the generalized coordinates; and $\mathbf{Q}(t)$ represents the n_0 vector of the non-stationary, zero-mean system excitation, whose evolutionary power spectrum matrix is denoted as $\mathbf{S}^{\mathbf{QQ}}(\omega, t)$. Further, $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ denote the $n_0 \times n_0$ mass, damping and stiffness matrices of the system. Next, adopting a redundant coordinates modeling, a new coordinates ℓ vector $\mathbf{x}$ ($\ell > n_0$), is considered. Thus, the mass, damping and stiffness $\ell \times \ell$ matrices are given by $\mathbf{M_x}$, $\mathbf{C_x}$ and $\mathbf{K_x}$, whereas $\mathbf{Q_x}$ denotes the corresponding ℓ vector of the system excitation. Considering additional constraints equations, Eq. (19) is recast in the form

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{20}$$

where $\bar{\mathbf{M}}_{\mathbf{x}}$, $\bar{\mathbf{C}}_{\mathbf{x}}$ and $\bar{\mathbf{K}}_{\mathbf{x}}$ denote the augmented mass, damping and stiffness $(m_0 + \ell) \times \ell$ matrices, given by

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{M}_{\mathbf{x}} \\ \mathbf{A} \end{bmatrix}, \tag{21}$$

$$\bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{C}_{\mathbf{x}} \\ \mathbf{E} \end{bmatrix} \tag{22}$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{K}_{\mathbf{x}} \\ \mathbf{L} \end{bmatrix}, \tag{23}$$

respectively; whereas $\bar{\mathbf{Q}}_{\mathbf{x}}$ represents the augmented excitation $(m_0 + \ell)$ vector, given by

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{Q}_{\mathbf{x}} \\ \mathbf{F} \end{bmatrix}. \tag{24}$$

Further, the $m_0 \times \ell$ matrices $\mathbf{A}, \mathbf{E}$ and $\mathbf{L}$ in Eqs. (21-24) pertain to the system constraints equation

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{25}$$

where

$$\mathbf{b} = \mathbf{F} - \mathbf{E}\dot{\mathbf{x}} - \mathbf{L}\mathbf{x}; \tag{26}$$

details on the derivation of Eqs. (20-26) can be found in Refs. (27; 28; 29).

Next, focusing on the joint time-frequency response characterization of the system of Eq. (20), it is assumed that the system excitation and corresponding response are modeled via the LSW representation of Eq. (8). Thus, the augmented system of Eq. (20) becomes

$$\bar{\mathbf{M}}_{\mathbf{x}} \sum_{(m,n)} \sum_{k} \ddot{\mathbf{x}}_{(m,n),k}(t) + \bar{\mathbf{C}}_{\mathbf{x}} \sum_{(m,n)} \sum_{k} \dot{\mathbf{x}}_{(m,n),k}(t) + \bar{\mathbf{K}}_{\mathbf{x}} \sum_{(m,n)} \sum_{k} \mathbf{x}_{(m,n),k}(t) = \sum_{(m,n)} \sum_{k} \bar{\mathbf{Q}}_{\mathbf{x},(m,n),k}(t). \tag{27}$$

The localized in time and frequency domains ℓ vector process $\mathbf{x}_{(m,n),k}(t)$ and $(m_0 + \ell)$ vector process $\bar{\mathbf{Q}}_{\mathbf{x},(m,n),k}(t)$ of the double summations in Eq. (27), are defined in Eq. (9) as combinations of localized monochromatic functions, i.e.,

$$\mathbf{x}_{(m,n),k}(t) = \bar{\mathbf{a}}_{(m,n),k} \cos\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right] + \bar{\mathbf{b}}_{(m,n),k} \sin\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right] \tag{28}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x},(m,n),k}(t) = \mathbf{e}_{(m,n),k} \cos\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right] + \mathbf{f}_{(m,n),k} \sin\left[\omega_{c,(m,n),k}\left(t - \frac{kT_0}{n-m}\right)\right]. \tag{29}$$

The terms $\bar{\mathbf{a}}_{(m,n),k}, \bar{\mathbf{b}}_{(m,n),k}$ and $\mathbf{e}_{(m,n),k}, \mathbf{f}_{(m,n),k}$ in Eqs. (28-29) correspond to statistically independent zero-mean random $(m_0 + \ell)$ vectors, whose variance is given by Eq. (10). Substituting Eqs. (28-29) in Eq. (27), and manipulating, yields

$$\begin{aligned} \omega^4_{c,(m,n),k} \bar{\mathbf{M}}_{\mathbf{x}} \mathbf{S}^{\mathbf{xx}}_{(m,n),k} \bar{\mathbf{M}}^{\mathrm{T}}_{\mathbf{x}} + \omega^2_{c,(m,n),k} \bar{\mathbf{C}}_{\mathbf{x}} \mathbf{S}^{\mathbf{xx}}_{(m,n),k} \bar{\mathbf{C}}^{\mathrm{T}}_{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}} \mathbf{S}^{\mathbf{xx}}_{(m,n),k} \bar{\mathbf{K}}^{\mathrm{T}}_{\mathbf{x}} \\ - \omega^2_{c,(m,n),k} \bar{\mathbf{M}}_{\mathbf{x}} \mathbf{S}^{\mathbf{xx}}_{(m,n),k} \bar{\mathbf{K}}^{\mathrm{T}}_{\mathbf{x}} - \omega^2_{c,(m,n),k} \bar{\mathbf{K}}_{\mathbf{x}} \mathbf{S}^{\mathbf{xx}}_{(m,n),k} \bar{\mathbf{M}}^{\mathrm{T}}_{\mathbf{x}} = \mathbf{S}^{\bar{\mathbf{Q}}_x \bar{\mathbf{Q}}_x}_{(m,n),k}, \end{aligned} \tag{30}$$

where $\mathbf{S}^{\bar{\mathbf{Q}}_x\bar{\mathbf{Q}}_x}_{(m,n),k}$ denotes the excitation EPS $(m_0+\ell)\times(m_0+\ell)$ matrix, and $\mathbf{S}^{\mathbf{xx}}_{(m,n),k}$ denotes the corresponding response EPS $\ell\times\ell$ matrix, both defined at different frequency and time bands. Further, manipulating Eq. (30), the expression

$$\mathbf{R}_{\mathbf{x}}\mathbf{S}^{\mathbf{xx}}_{(m,n),k}\left[-\omega^2_{c,(m,n),k}\bar{\mathbf{M}}^{\mathrm{T}}_{\mathbf{x}}-i\omega_{c,(m,n),k}\bar{\mathbf{C}}^{\mathrm{T}}_{\mathbf{x}}+\bar{\mathbf{K}}^{\mathrm{T}}_{\mathbf{x}}\right]=\mathbf{S}^{\bar{\mathbf{Q}}_x\bar{\mathbf{Q}}_x}_{(m,n),k}, \tag{31}$$

is derived, where the $(m_0+\ell)\times\ell$ matrix $\mathbf{R}_{\mathbf{x}}$ has the form

$$\mathbf{R}_{\mathbf{x}}=\left[-\omega^2_{c,(m,n),k}\bar{\mathbf{M}}_{\mathbf{x}}+\mathrm{i}\omega_{c,(m,n),k}\bar{\mathbf{C}}_{\mathbf{x}}+\bar{\mathbf{K}}_{\mathbf{x}}\right]. \tag{32}$$

In Eq. (32), $\bar{\mathbf{M}}_{\mathbf{x}}$, $\bar{\mathbf{C}}_{\mathbf{x}}$ and $\bar{\mathbf{K}}_{\mathbf{x}}$ denote the $(m_0+\ell)\times\ell$ augmented mass, damping and stiffness matrices of Eqs. (21-23), whereas $\omega_{c,(m,n),k}$ represents the central frequency of Eq. (6).

Note, in passing, that using the minimum number of generalized coordinates for the formulation of the system governing equations of motion (see Eq. (19)), results in non-singular mass, damping and stiffness matrices. This, in turn, facilitates the derivation of a closed form, wavelet-coefficient-based excitation-response relationship. Specifically, for the system of Eq. (19) it holds (13)

$$\mathbf{S}^{\mathbf{qq}}_{(m,n),k}=\mathbf{H}[\omega_{c,(m,n),k}]\mathbf{S}^{\mathbf{QQ}}_{(m,n),k}\mathbf{H}[\omega_{c,(m,n),k}]^{\mathrm{T}*}, \tag{33}$$

where

$$\mathbf{H}[\omega_{c,(m,n),k}]=\left[-\omega^2_{c,(m,n),k}\mathbf{M}+\mathrm{i}\omega_{c,(m,n),k}\mathbf{C}+\mathbf{K}\right]^{-1} \tag{34}$$

is the localized harmonic-wavelet-based, frequency response function (HW-FRF). Clearly, the definition of the HW-FRF relies on the existence of the inverse matrix of Eq. (34). Thus, it is readily seen that following a redundant DOFs modeling approach, the singular mass, damping and stiffness matrices of Eqs. (21-23), hinder the derivation of the corresponding HW-FRF. However, employing elements of M-P generalized matrix inverse theory, a generalized HW-FRF is proposed herein to account for the case of systems with singular matrices.

In this regard, utilizing Eq. (3), the solution to Eq. (31) takes the form

$$\mathbf{S}^{\mathbf{xx}}_{(m,n),k}\left[-\omega^2_{c,(m,n),k}\bar{\mathbf{M}}^{\mathrm{T}}_{\mathbf{x}}-i\omega_{c,(m,n),k}\bar{\mathbf{C}}^{\mathrm{T}}_{\mathbf{x}}+\bar{\mathbf{K}}^{\mathrm{T}}_{\mathbf{x}}\right]=\mathbf{R}^{+}_{\mathbf{x}}\mathbf{S}^{\bar{\mathbf{Q}}_x\bar{\mathbf{Q}}_x}_{(m,n),k}+(\mathbf{I}_\ell-\mathbf{R}^{+}_{\mathbf{x}}\mathbf{R}_{\mathbf{x}})\mathbf{Y}, \tag{35}$$

where $\mathbf{R}^{+}_{\mathbf{x}}$ denotes the $\ell\times(m_0+\ell)$ M-P inverse of $\mathbf{R}_{\mathbf{x}}$, and $\mathbf{Y}$ is an arbitrary $\ell\times(m_0+\ell)$ matrix. Clearly, the arbitrary matrix $\mathbf{Y}$ implies a family of solutions for Eq. (35), instead of a unique solution. Nevertheless, the rank of matrix $\mathbf{R}_{\mathbf{x}}$ facilitates the selection of a unique solution. Specifically, when $\mathbf{R}_{\mathbf{x}}$ has full column rank, i.e., it has linearly independent columns, its M-P inverse takes the form (32)

$$\mathbf{R}^{+}_{\mathbf{x}}=(\mathbf{R}^{*}_{\mathbf{x}}\mathbf{R}_{\mathbf{x}})^{-1}\mathbf{R}^{*}_{\mathbf{x}}, \tag{36}$$

where $\mathbf{R}^{*}_{\mathbf{x}}$ corresponds to the conjugate matrix of $\mathbf{R}_{\mathbf{x}}$. Thus, taking into account Eq. (36), Eq. (35) is equivalently written as

$$\mathbf{S}^{\mathbf{xx}}_{(m,n),k}\left[-\omega^2_{c,(m,n),k}\bar{\mathbf{M}}^{\mathrm{T}}_{\mathbf{x}}-i\omega_{c,(m,n),k}\bar{\mathbf{C}}^{\mathrm{T}}_{\mathbf{x}}+\bar{\mathbf{K}}^{\mathrm{T}}_{\mathbf{x}}\right]=\mathbf{R}^{+}_{\mathbf{x}}\mathbf{S}^{\bar{\mathbf{Q}}_x\bar{\mathbf{Q}}_x}_{(m,n),k}. \tag{37}$$

Further, applying the conjugate and transpose operations on both sides of Eq. (32) implies

$$\left[-\omega^2_{c,(m,n),k}\bar{\mathbf{M}}^{\mathrm{T}}_{\mathbf{x}}-i\omega_{c,(m,n),k}\bar{\mathbf{C}}^{\mathrm{T}}_{\mathbf{x}}+\bar{\mathbf{K}}^{\mathrm{T}}_{\mathbf{x}}\right]=\mathbf{R}^{*\mathrm{T}}_{\mathbf{x}}. \tag{38}$$

Also, noting that for full column rank matrix $\mathbf{R}_{\mathbf{x}}$, its conjugate transpose $\mathbf{R}^{*\mathrm{T}}_{\mathbf{x}}$ has full row rank, i.e., its rows are linearly independent, results in (32)

$$\mathbf{R}^{*\mathrm{T}}_{\mathbf{x}}\left(\mathbf{R}^{*\mathrm{T}}_{\mathbf{x}}\right)^{+}=\mathbf{I}_{m+\ell}. \tag{39}$$

Therefore, taking into account Eqs. (38) and (39), and also denoting by $\boldsymbol{\alpha}_{\mathbf{x}}[\omega_{c,(m,n),k}]$ the M-P inverse of matrix $\mathbf{R}_{\mathbf{x}}$ of Eq. (32), i.e.,

$$\boldsymbol{\alpha}_{\mathbf{x}}[\omega_{c,(m,n),k}]=\left[-\omega^2_{c,(m,n),k}\bar{\mathbf{M}}_{\mathbf{x}}+\mathrm{i}\omega_{c,(m,n),k}\bar{\mathbf{C}}_{\mathbf{x}}+\bar{\mathbf{K}}_{\mathbf{x}}\right]^{+}, \tag{40}$$

Eq. (37) becomes

$$\mathbf{S}^{\mathbf{xx}}_{(m,n),k}=\boldsymbol{\alpha}_{\mathbf{x}}[\omega_{c,(m,n),k}]\mathbf{S}^{\bar{\mathbf{Q}}_x\bar{\mathbf{Q}}_x}_{(m,n),k}\left(\boldsymbol{\alpha}_{\mathbf{x}}[\omega_{c,(m,n),k}]\right)^{\mathrm{T}*}. \tag{41}$$

Matrix $\boldsymbol{\alpha}_{\mathbf{x}}[\omega_{c,(m,n),k}]$ of Eq. (40) represents a generalization of the localized in time and frequency HW-FRF matrix $\mathbf{H}\left[\omega_{c,(m,n),k}\right]$ (see Eq. (34)). In this regard, the herein proposed approach can be construed as a generalization of the results in Ref. (13), for deriving the response evolutionary power spectrum (EPS) of linear MDOF systems with singular matrices.

3.2 Nonlinear systems with singular matrices

In this section, aiming at the response EPS determination of MDOF chain-like nonlinear structural systems with singular matrices, a recently proposed, harmonic-wavelet-based version of the statistical linearization methodology (9; 13) is extended. Statistical linearization constitutes one of the most versatile approximate techniques for nonlinear system response determination and/or characterization (2; 39), with a wide variety of applications over the last decades. It has been successfully adapted and/or extended for application in conjunction with fractional calculus (40; 41; 10) and wavelet-based solutions frameworks (9), among others. The method is applied in two steps, which are summarized as follows. First, the original nonlinear system is replaced with an equivalent linear one, and then, the error between the two systems is formed and minimized (2). Based on the fact that solution frameworks for treating the equivalent linear system are readily available, the rationale behind the method is that the latter can be used as approximations to the solution of the original nonlinear system.

In this regard, adopting a redundant coordinates modeling (27; 28), the equations of motion for the nonlinear version of the system of Eq. (20) takes the form

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}} + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x} + \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}) = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{42}$$

where the augmented mass, damping, stiffness matrices and excitation vector are given by Eqs. (21-23) and Eq. (24), respectively. Further, the augmented nonlinear $(m_0 + \ell)$ vector of the system, which depends on the response displacement $\mathbf{x}$ and the response velocity $\dot{\mathbf{x}}$, is given by (28)

$$\bar{\mathbf{\Phi}}_{\mathbf{x}} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{\Phi}_{\mathbf{x}} \\ \mathbf{0} \end{bmatrix}. \tag{43}$$

In Eq. (43), $\mathbf{\Phi}_{\mathbf{x}}$ denotes the nonlinear ℓ vector of the system governing equations of motion, due to adopting a redundant coordinates modeling (28).

Next, considering the LSW representation of Eq. (8) for the system excitation $\mathbf{x}(t)$, and thus assuming that $\mathbf{x}(t)$ is represented by the sum of its wavelet coefficients, an equivalent to Eq. (42) linear MDOF system is defined as

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}} + \sum_{(m,n)}\sum_{k} \bar{\mathbf{C}}_{eq,(m,n),k}\dot{\mathbf{x}}_{(m,n),k} + \sum_{(m,n)}\sum_{k} \bar{\mathbf{K}}_{eq,(m,n),k}\mathbf{x}_{(m,n),k} = \bar{\mathbf{Q}}_{\mathbf{x}}(t). \tag{44}$$

In Eq. (44), $\bar{\mathbf{C}}_{eq,(m,n),k}$ and $\bar{\mathbf{K}}_{eq,(m,n),k}$ denote the $(m_0 + \ell) \times \ell$ equivalent linear damping and stiffness matrices, which account for the nonlinearity of the nonlinear system of Eq. (42), and are both time and frequency dependent. Continuing with the application of the statistical linearization method, the error function ε is formed as the difference between the original nonlinear system of Eq. (42), and its linear equivalent Eq. (44), i.e.,

$$\begin{aligned} \varepsilon \;=\; & \bar{\mathbf{\Phi}}_{\mathbf{x}}\left(\sum_{(m,n)}\sum_{k}\mathbf{x}_{(m,n),k}, \sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k}\right) + \bar{\mathbf{C}}_{\mathbf{x}}\sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k} + \bar{\mathbf{K}}_{\mathbf{x}}\sum_{(m,n)}\sum_{k}\mathbf{x}_{(m,n),k} \\ & - \sum_{(m,n)}\sum_{k}\bar{\mathbf{C}}_{eq,(m,n),k}\dot{\mathbf{x}}_{(m,n),k} - \sum_{(m,n)}\sum_{k}\bar{\mathbf{K}}_{eq,(m,n),k}\mathbf{x}_{(m,n),k}. \end{aligned} \tag{45}$$

It is noted that for the formulation of Eq. (45), Eq. (42) is also expressed in terms of the LSW representation of Eq. (8). Then, the error function Eq. (45) is minimized by considering the orthogonality properties of the monochromatic functions at a given frequency band and time location (see Eqs. (11-12)). Specifically, Eq. (45) is first post-multiplied by the transpose of the response displacement vector $\mathbf{x}_{(m,n),k}$. Subsequently, integrating with respect to time and ensemble averaging results in zero average error, i.e.,

$$\begin{aligned} \mathbb{E}\Bigg[\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} & \bar{\mathbf{\Phi}}_{\mathbf{x}}\left(\sum_{(m,n)}\sum_{k}\mathbf{x}_{(m,n),k}, \sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k}\right)\mathbf{x}^{\mathrm{T}}_{(m,n),k}\mathrm{d}t \\ & +(\bar{\mathbf{C}}_{\mathbf{x}} - \bar{\mathbf{C}}_{eq,(m,n),k})\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} \dot{\mathbf{x}}_{(m,n),k}\mathbf{x}^{\mathrm{T}}_{(m,n),k}\mathrm{d}t \\ & +(\bar{\mathbf{K}}_{\mathbf{x}} - \bar{\mathbf{K}}_{eq,(m,n),k})\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} \mathbf{x}_{(m,n),k}\mathbf{x}^{\mathrm{T}}_{(m,n),k}\mathrm{d}t \Bigg] \;=\; \mathbf{0}. \end{aligned} \tag{46}$$

In a similar manner, Eq. (45) also yields

$$\begin{aligned}\mathbb{E}\Bigg[\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} \bar{\mathbf{\Phi}}_{\mathbf{x}}\left(\sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k},\sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k}\right)\dot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}\mathrm{d}t \\ +(\bar{\mathbf{C}}_{\mathbf{x}}-\bar{\mathbf{C}}_{eq,(m,n),k})\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}}\dot{\mathbf{x}}_{(m,n),k}\dot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k} \\ +(\bar{\mathbf{K}}_{\mathbf{x}}-\bar{\mathbf{K}}_{eq,(m,n),k})\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}}\mathbf{x}_{(m,n),k}\dot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}\Bigg] &= \mathbf{0}.\end{aligned} \tag{47}$$

Further, Eqs. (46) and (47) can be equivalently written as

$$\begin{aligned}\mathbb{E}\Bigg[\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} \bar{\mathbf{\Phi}}_{\mathbf{x}}\left(\sum_{(m,n)}\sum_{k}\mathbf{x}_{(m,n),k},\sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k}\right)\mathbf{x}^{\mathrm{T}}_{(m,n),k}\mathrm{d}t\Bigg] \\ +4\pi(\bar{\mathbf{K}}_{\mathbf{x}}-\bar{\mathbf{K}}_{eq,(m_i,n_i),k})\mathbf{S}^{\mathbf{xx}}_{(m_i,n_i),k} &= \mathbf{0}\end{aligned} \tag{48}$$

and

$$\begin{aligned}\mathbb{E}\Bigg[\int_{\frac{kT_0}{n-m}}^{\frac{(k+1)T_0}{n-m}} \bar{\mathbf{\Phi}}_{\mathbf{x}}\left(\sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k},\sum_{(m,n)}\sum_{k}\dot{\mathbf{x}}_{(m,n),k}\right)\dot{\mathbf{x}}^{\mathrm{T}}_{(m,n),k}\mathrm{d}t\Bigg] \\ +4\pi(\bar{\mathbf{C}}_{\mathbf{x}}-\bar{\mathbf{C}}_{eq,(m_i,n_i),k})\mathbf{S}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k} &= \mathbf{0}.\end{aligned} \tag{49}$$

Eqs. (48) and (49) connect, in essence, the localized in frequency and time intervals $m\Delta\omega \le \omega < n\Delta\omega$ and $\frac{kT_0}{n-m} \le t < \frac{(k+1)T_0}{n-m}$, response EPS $\mathbf{S}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$ and $\mathbf{S}^{\mathbf{xx}}_{(m_i,n_i),k}$, with the corresponding localized equivalent linear elements $\bar{\mathbf{C}}_{eq,(m,n),k}$ and $\bar{\mathbf{K}}_{eq,(m,n),k}$. Furthermore, taking into account Eqs. (13-18), $\mathbf{S}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$ in Eq. (49) can be replaced by $\mathbf{S}^{\mathbf{xx}}_{(m_i,n_i),k}$. In this regard, Eqs. (48) and (49), together with Eq. (41), form a coupled system of nonlinear equations to be solved for determining $\bar{\mathbf{C}}_{eq,(m,n),k}$ and $\bar{\mathbf{K}}_{eq,(m,n),k}$, and the response EPS $\mathbf{S}^{\mathbf{xx}}_{(m_i,n_i),k}$. For the solution of the coupled system of equations, the following iterative scheme can be applied (9; 13). First, initial values for the equivalent elements $\bar{\mathbf{C}}_{eq,(m,n),k}$ and $\bar{\mathbf{K}}_{eq,(m,n),k}$ are considered, and solving Eq. (41), initial values for $\mathbf{S}^{\mathbf{xx}}_{(m_i,n_i),k}$ are derived. Then, the latter is used in conjunction with Eqs. (48-49) for updating the values of the equivalent elements $\bar{\mathbf{C}}_{eq,(m,n),k}$ and $\bar{\mathbf{K}}_{eq,(m,n),k}$. The process repeats until convergence.

Next, considering the case of polynomial kind nonlinearities, the described statistical linearization methodology results in closed form solutions for determining the equivalent linear elements. This assumption is considered not only for facilitating the ensuing derivation of the closed form solutions, i.e., for simplicity reasons, but also due to its practical merit, as nonlinearities of this kind are often met in structural engineering applications (2; 13; 38). In this regard, considering a redundant coordinates modeling of the equations of motion given by Eq. (19), the cubic nonlinearity ℓ vector is written in the form

$$\mathbf{\Phi}_{\mathbf{x}} = \begin{bmatrix}\varepsilon_1 k_1 x_1^3 \\ \vdots \\ \varepsilon_\ell k_\ell x_\ell^3\end{bmatrix} + \begin{bmatrix}\lambda_1 c_1 \dot{x}_1^3 \\ \vdots \\ \lambda_\ell c_\ell \dot{x}_\ell^3\end{bmatrix}, \tag{50}$$

where ε_i and λ_i, for $i = 1, 2, \dots, \ell$, denote the magnitude of the nonlinearity for the stiffness and damping of the system, respectively.

Further, considering the augmented nonlinear vector $\bar{\mathbf{\Phi}}_{\mathbf{x}}$ of Eq. (43) in conjunction with Eqs. (48-49), closed form expressions are derived for the augmented equivalent $(m_0 + \ell) \times \ell$ elements $\bar{\mathbf{C}}_{eq,(m,n),k}$ and $\bar{\mathbf{K}}_{eq,(m,n),k}$. These are expressed in terms of summations of the response EPS over all (m_i, n_i) at a specific k, and for all pairs of (m, n) and k. Specifically, for the determination of the equivalent damping element, taking into account Eqs. (43) and (50), Eq. (49) implies

$$\mathbf{D}_d + (\bar{\mathbf{C}}_{\mathbf{x}} - \bar{\mathbf{C}}_{eq,(m_i,n_i),k})\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k} = \mathbf{0}, \tag{51}$$

where $\mathbf{D}_d$ is an $(m_0 + \ell) \times \ell$ matrix whose entries depend on the damping nonlinearity, as well as on the system constraints defined in Eq. (25); and $\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$ denotes the local auto/cross EPS $\ell \times \ell$ matrix of the response velocity process $\dot{\mathbf{x}}_{(m,n),k}$. It is noted that, in contrast to the standard linearization approach of Ref. (13) where the corresponding local auto/cross EPS matrix is diagonal, matrix $\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$ of Eq. (51) potentially has some non-zero off-diagonal entries. This is due to the redundant DOFs employed in the system modeling. Specifically, the dependence between the redundant DOFs which form the coordinates vector $\mathbf{x}$ implies linear dependence between some of the columns of matrix $\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$. Therefore, $\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$ is singular, and thus, in order to derive a closed form expression for the equivalent damping element, a special treatment is required for solving Eq. (51).

In this regard, taking into account Eq. (3), Eq. (51) implies

$$\bar{\mathbf{C}}_{eq,(m_i,n_i),k} = \mathbf{D}_d \left(\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}\right)^+ + \bar{\mathbf{C}}_{\mathbf{x}} + \left\{ \left((\mathbf{I}_\ell - \left((\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k})^{\mathrm{T}} \right)^+ (\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k})^{\mathrm{T}} \right) \mathbf{Y}_1 \right\}^{\mathrm{T}}, \tag{52}$$

where $\mathbf{Y}_1$ is a $(m_0 + \ell) \times \ell$ matrix of arbitrary elements. In a similar manner, taking into account Eqs. (43) and (50), Eq. (49) yields

$$\bar{\mathbf{K}}_{eq,(m_i,n_i),k} = \mathbf{D}_s \left(\tilde{\mathbf{S}}^{\mathbf{x}\mathbf{x}}_{(m_i,n_i),k}\right)^+ + \bar{\mathbf{K}}_{\mathbf{x}} + \left\{ \left((\mathbf{I}_\ell - \left((\tilde{\mathbf{S}}^{\mathbf{x}\mathbf{x}}_{(m_i,n_i),k})^{\mathrm{T}} \right)^+ (\tilde{\mathbf{S}}^{\mathbf{x}\mathbf{x}}_{(m_i,n_i),k})^{\mathrm{T}} \right) \mathbf{Y}_2 \right\}^{\mathrm{T}}. \tag{53}$$

In this case, $\mathbf{D}_s$ is an $(m_0 + \ell) \times \ell$ matrix whose entries depend on the stiffness nonlinearity and the constraints of Eq. (25), whereas $\tilde{\mathbf{S}}^{\mathbf{x}\mathbf{x}}_{(m_i,n_i),k}$ denotes the local auto/cross EPS $\ell \times \ell$ matrix of the response displacement $\mathbf{x}_{(m,n),k}$; $\mathbf{Y}_2$ is a $(m_0 + \ell) \times \ell$ matrix of arbitrary elements.

Clearly, due to the arbitrary matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$, Eqs. (52) and (53) form a family of solutions, while $\bar{\mathbf{C}}_{eq,(m,n),k}$ and $\bar{\mathbf{K}}_{eq,(m,n),k}$ correspond to the unique damping and stiffness matrices of the equivalent linear system defined in Eq. (44). However, taking into account that the M-P generalized matrix inverse framework employed in the derivation of Eqs. (52) and (53) corresponds, in essence, to the solution of a quadratic (least squares) optimization problem, it is feasible to select a unique solution for each of the equivalent elements. In this regard, the intuitively simplest solution among the family of solutions, which also coincides with the minimal mean square solution of the quadratic problem above (32), is considered. Thus, setting the arbitrary matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$ equal to null matrix, Eqs. (52) and (53) become

$$\bar{\mathbf{C}}_{eq,(m_i,n_i),k} = \mathbf{D}_d \left(\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}\right)^+ + \bar{\mathbf{C}}_{\mathbf{x}} \tag{54}$$

and

$$\bar{\mathbf{K}}_{eq,(m_i,n_i),k} = \mathbf{D}_s \left(\tilde{\mathbf{S}}^{\mathbf{x}\mathbf{x}}_{(m_i,n_i),k}\right)^+ + \bar{\mathbf{K}}_{\mathbf{x}}, \tag{55}$$

respectively. Clearly, Eqs. (54) and (55) define a unique solution for the equivalent damping and stiffness elements, respectively. Further, they constitute a generalization of the corresponding results in Ref. (13), to account for systems with singular matrices and polynomial kind nonlinearities, subjected to non-stationary excitation.

4 Numerical examples

4.1 Non-stationary stochastic excitation

The considered in the ensuing numerical examples MDOF systems, are subjected to non-stationary stochastic excitation described by non-separable, evolutionary power spectra of the form

$$S(\omega, t) = S_0 \left(\frac{\omega}{5\pi}\right)^2 \exp(-c_0 t) t^2 \exp\left(-\left(\frac{\omega}{5\pi}\right)^2 t\right), \tag{56}$$

where $S_0, c_0 \in \mathbb{R}$. Eq. (56), firstly introduced in Ref. (42), is used as an indicative seismic excitation, since it encloses the main "build-up" and "die-off" characteristics of the ground motion, while its dominant frequency decreases with time (43; 44; 10). The spectral representation method is employed for deriving compatible to the EPS of Eq. (56) realizations ((45)) and the response EPS is obtained by averaging the mean square magnitude of the corresponding wavelet coefficients (9), i.e.,

$$S(\omega, t_k) = \frac{T_0}{2\pi(n-m)} \mathrm{E}\left[\mid W^G_{(m,n),k} \mid^2 \right]. \tag{57}$$

The parameter value $n - m = 5$ is used in the ensuing analysis. Moreover, the Mean Instantaneous Frequency (MIF) given by

$$\mathrm{MIF}(t) = \frac{\int_\omega \omega S(t,\omega) d\omega}{\int_\omega S(t,\omega) d\omega} \tag{58}$$

is included in the ensuing analysis for capturing the evolution of the "effective instantaneous frequency" for the non-stationary system response (46).

4.2 Linear systems with singular matrices

For the assessment of the herein proposed solution framework, the EPS for each DOF of the 3−DOF linear system depicted in Fig. 1, is determined. The system consists of three masses m_1, m_2 and m_3 interconnected with linear springs and dampers. In particular, mass m_1 is connected to the foundation by a linear spring and a linear damper with stiffness and damping coefficients k_1 and c_1, respectively, and to masses m_2 and m_3 by linear springs with stiffness coefficients k_2 and k_4. Further, mass m_2 is connected to mass m_3 by a linear spring of stiffness coefficient k_3 and a linear damper of damping coefficient c_2. The system is subjected to random force $Q_1(t)$ which is applied on mass m_1, and to random force $Q_3(t)$ applied on mass m_3, which are both described by the EPS of Eq. (56).

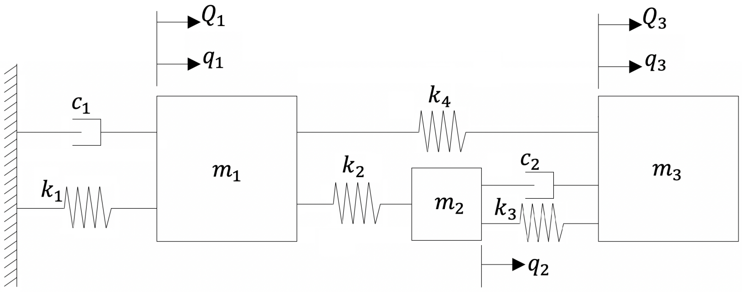


Fig. 1: Three degree-of-freedom system subjected to non-stationary stochastic excitation.

Following, a standard Newtonian approach for the formulation of the system governing equations of motion, and considering the (generalized) coordinates vector $\mathbf{q}^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$, the mass, damping and stiffness matrices are given by

$$\mathbf{M_q} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C_q} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix} \tag{59}$$

and

$$\mathbf{K_q} = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix}, \tag{60}$$

respectively. Next, considering a relative displacement modeling, the coordinates vector $\mathbf{y}^{\mathrm{T}} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$ is defined. Vector $\mathbf{y}$ denotes the relative displacement between the adjacent DOFs (2), i.e.,

$$y_1 = q_1, \; y_2 = q_2 - q_1, \; y_3 = q_3 - q_2. \tag{61}$$

Thus, the mass, damping and stiffness matrices in the relative coordinates system take the form

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ m_2 & m_2 & 0 \\ m_3 & m_3 & m_3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & -c_2 \\ 0 & 0 & c_2 \end{bmatrix} \tag{62}$$

and

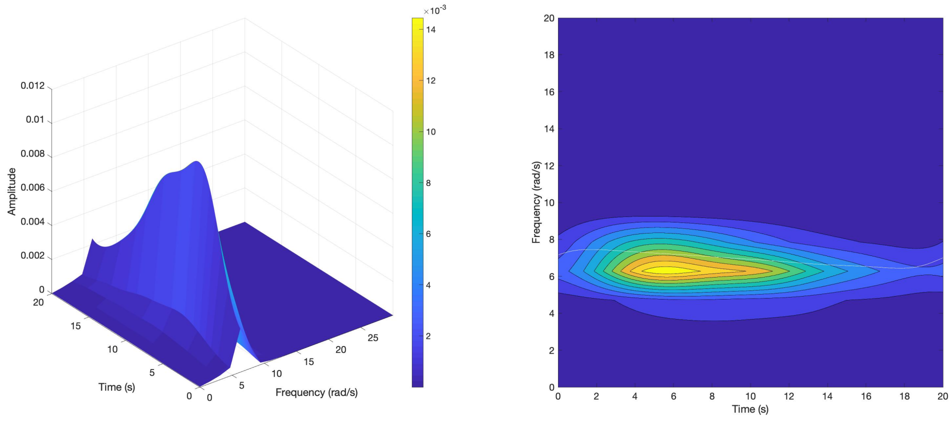
$$\mathbf{K} = \begin{bmatrix} k_1 & -k_2 - k_4 & -k_4 \\ 0 & k_2 & -k_3 \\ 0 & k_4 & k_3 + k_4 \end{bmatrix}, \tag{63}$$

respectively. Finally, the excitation vector $\mathbf{Q}(t)$ is given by

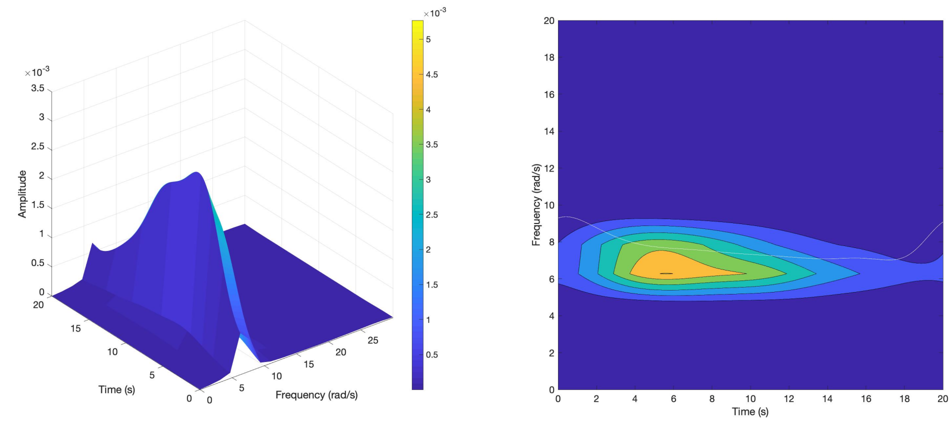
$$\mathbf{Q} = \begin{bmatrix} Q_1(t) \\ 0 \\ Q_3(t) \end{bmatrix}, \tag{64}$$

where $Q_1(t) = Q_3(t) = Q(t)$. Further, assuming the system and excitation parameter values $m_1 = m_2 = m_3 = 1$, $c_1 = c_2 = 4.3$, $c_3 = 1.4$, $k_1, k_2, k_3 = 256$, $k_4 = 64$ and $S_0 = 10$, $c_0 = 0.15$, and resorting to Eq. (33-34), the generalized response spectra are estimated. The response EPS for each DOF of the system of Fig. 1 is depicted in Figs. 2(a)-2(c). Also, the MIF of Eq. (58) is included for each DOF.

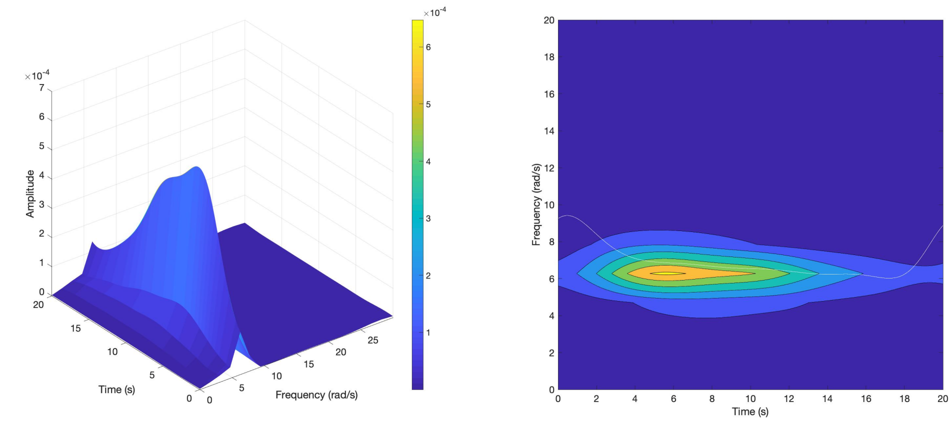
Next, the herein proposed methodology for deriving the EPS of the system response is applied to the system of Fig. 1. In this regard, adopting a redundant coordinates modeling of the equations of motion, the 3−DOF system is



(a)



(b)



(c)

Fig. 2: Response EPS of the linear system of Fig. 1, subject to non-stationary excitation ($S_0 = 10, c_0 = 0.15$). (a) 1st DOF y_1 of the system. (b) 2nd DOF y_2 of the system. (c) 3rd DOF y_3 of the system.

decomposed into its component subsystems, as shown in Fig. 3. This is attained by considering the coordinates vector

$$\mathbf{x} = \begin{bmatrix} \bar{x}_1 \\ x_2 \\ \bar{x}_3 \\ x_4 \\ \bar{x}_5 \end{bmatrix}, \tag{65}$$

where $\bar{x}_1, \bar{x}_3$ and $\bar{x}_5$ denote the displacements of masses m_1, m_2 and m_3, respectively; and x_2, x_4 correspond to the redundant DOFs which, in essence, account for the constraints connecting the partial subsystems (see Fig. 3). It holds

$$x_1 + d = x_2 \tag{66}$$

and

$$x_2 + x_3 + d = x_4, \tag{67}$$

where d is the physical length of each mass $m_i, i = 1, 2, 3$. Taking into account the geometry of the system in Fig. 3, as well as Eqs. (66-67), the expressions

$$\bar{x}_1 + l_{1,0} + d = x_2 \tag{68}$$

and

$$x_2 + \bar{x}_3 + l_{3,0} + d = x_4, \tag{69}$$

where l_i is the unstretched length of the spring for the mass m_i ($i = 1, 2, 3$), form a set of equations which connect the system constraints with the redundant coordinates. Specifically, differentiating twice with respect to time Eqs. (68)-(69), the 2×5 matrix $\mathbf{A}$ of Eq. (25) and the $2-$vector $\mathbf{b}$ of Eq. (26) take the form

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \tag{70}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{71}$$

respectively.

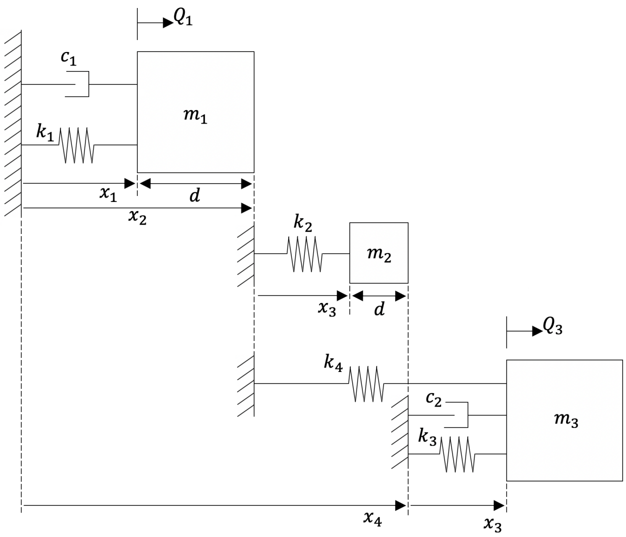


Fig. 3: Modeling the three degree-of-freedom system of Fig. 1 by using redundant coordinates.

In this regard, the equations of motion are formed, and the augmented mass, damping and stiffness 7×5 matrices of Eqs. (21-23) become

$$\bar{\mathbf{M}}_\mathbf{x} = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.2 & 0.2 & 0.2 \\ -0.2 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.2 & 0.6 & 0.6 & 0.6 & 0.6 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \bar{\mathbf{C}}_\mathbf{x} = \begin{bmatrix} 1.72 & 0 & 0 & 0 & 0 \\ 1.72 & 0 & 0 & 0 & 0 \\ -0.86 & 0 & 0 & 0 & 0 \\ 0.86 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.30 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{72}$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 102.4 & 12.8 & -51.2 & -12.8 & -12.8 \\ 102.4 & 12.8 & -51.2 & -12.8 & -12.8 \\ -51.2 & -38.4 & 153.6 & 38.4 & 38.4 \\ 51.2 & -25.6 & 102.4 & 25.6 & 25.6 \\ 0 & -64 & 0 & 64 & 320 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{73}$$

respectively; whereas the augmented excitation vector of Eq. (24) takes the form

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} 0.6Q(t) \\ 0.6Q(t) \\ 0.2Q(t) \\ 0.8Q(t) \\ Q(t) \\ 0 \\ 0 \end{bmatrix}. \tag{74}$$

Next, the 7×5 matrix $\mathbf{R}_{\mathbf{x}}$ is formed, for which $\operatorname{rank}(\mathbf{R}_{\mathbf{x}}) = 5$. Therefore, utilizing the input-output expression of Eq. (41), and taking into account the M-P HW-FRF of Eq. (40), the response EPS for $\bar{x}_1$, $\bar{x}_3$ and $\bar{x}_5$ of the equivalent linear system, is obtained in an analytical manner. The corresponding results, along with the MIF of Eq. (58), are plotted in Figs. 4(a), 4(b) and 4(c). Comparing Fig. 4(a) with Fig. 2(a), Fig. 4(b) with Fig. 2(b), and Fig. 4(c) with Fig. 2(c), it is seen that the results obtained by applying the herein proposed M-P inverse framework are in agreement with those obtained by following the standard formulation. This is also deduced by Fig. 5, where the response EPS for each DOF of the two systems (Fig. 1 and Fig. 3) is determined for both formulations, and is plotted at different time instants ($t = 5.8s$ and $t = 12s$). In this regard, it can be argued that the herein proposed approach constitutes a generalized matrix inverses extension of the harmonic-wavelet-based technique of Ref. (13).

4.3 Nonlinear systems with singular matrices

The $2-$DOF chain-like structural nonlinear system of rigid masses m_1 and m_2 that is shown in Fig. 6 is considered for demonstrating the efficiency of the linearization scheme. Mass m_1 is connected to the foundation by a nonlinear spring of linear-plus-cubic type, and a nonlinear damper of the same nonlinearity type. Further, mass m_1 is connected to mass m_2 by corresponding linear spring and damper with stiffness and damping coefficients k_2 and c_2, respectively. The nonlinearity magnitude of the nonlinear spring is denoted by ε_1, whereas λ_1 denotes the nonlinearity magnitude of the nonlinear damper. The system is subjected to non-stationary stochastic excitations $\mathbf{Q}_1(t)$ and $\mathbf{Q}_2(t)$, both described by a non-separable EPS of the form given by Eq. (56).

Next, the system governing equations of motion are derived. In this regard, the generalized coordinates y_1, y_2 are utilized, and relative displacements are introduced for facilitating the ensuing analysis. Thus, the mass, damping and stiffness matrices for the system of Fig. 6 are given by (2)

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ m_2 & m_2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} c_1 & -c_2 \\ 0 & c_2 \end{bmatrix} \tag{75}$$

and

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_2 \\ 0 & k_2 \end{bmatrix}, \tag{76}$$

respectively. Further, the system nonlinearity is written in the vector form

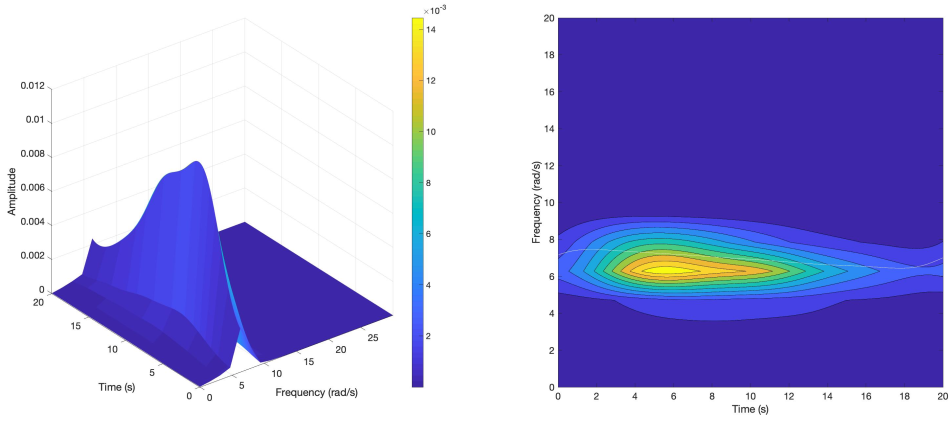
$$\mathbf{\Phi} = \begin{bmatrix} \varepsilon_1 k_1 y_1^3 + \lambda_1 c_1 \dot{y}_1^3 \\ 0 \end{bmatrix} \tag{77}$$

and the system excitation is

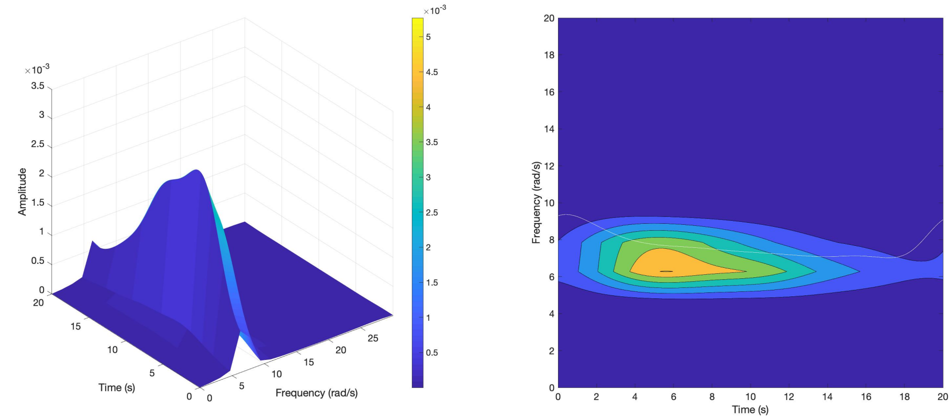
$$\mathbf{Q}(t) = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{78}$$

Then, the standard linearization approach of Ref. (13) is followed for determining the response EPS. In this regard, the equivalent linear elements $\mathbf{C}_{eq,(m_i,n_i),k}$ and $\mathbf{K}_{eq,(m_i,n_i),k}$ are given by

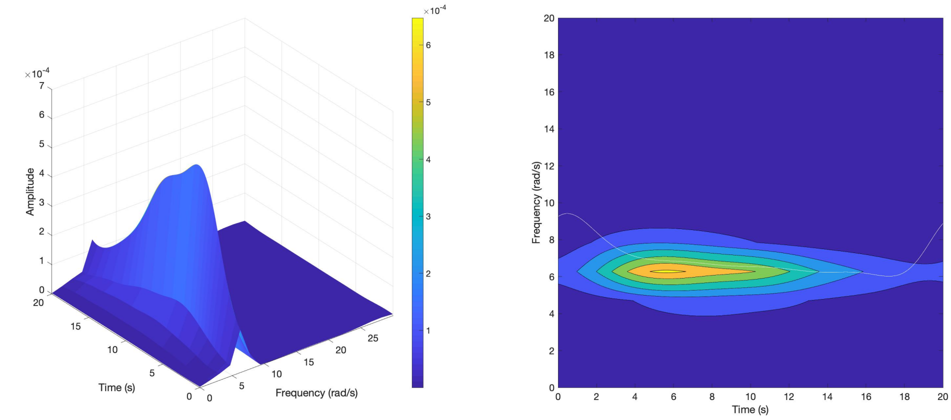
$$\mathbf{C}_{eq,(m_i,n_i),k} = 6(n-m)\Delta\omega \begin{bmatrix} \lambda_1 c_1 \sum_j S^{\dot{y}_1\dot{y}_1}_{(m_j,n_j),k} & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{C} \tag{79}$$



(a)

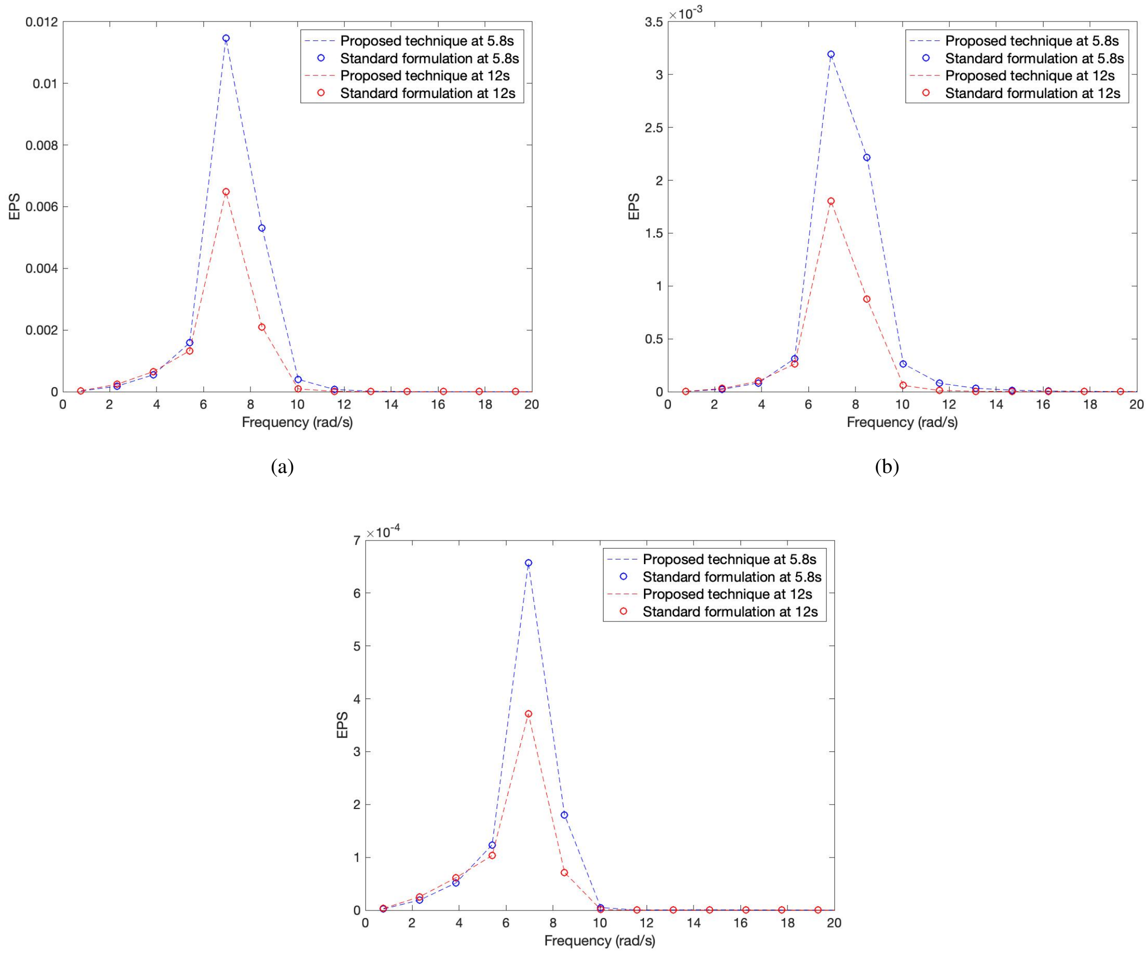


(b)



(c)

Fig. 4: Response EPS of the linear system of Fig. 3, subject to non-stationary excitation ($S_0 = 10, c_0 = 0.15$). (a) 1st DOF $\bar{x}_1$ of the system. (b) 3rd DOF $\bar{x}_3$ of the system. (c) 5th DOF $\bar{x}_5$ of the system.



(a) (b) (c)

Fig. 5: Response EPS of the linear system of Fig. 3 at time instants $t = 5.8s$ and $t = 12s$, subject to non-stationary stochastic excitation ($S_0 = 10, c_0 = 0.15$). Comparison between standard formulation and the proposed technique. (a) 1^{st} DOF (y_1 vis-à-vis $\bar{x}_1$); (b) 2^{nd} DOF (y_2 vis-à-vis $\bar{x}_3$); (c) 3^{rd} DOF (y_3 vis-à-vis $\bar{x}_5$).

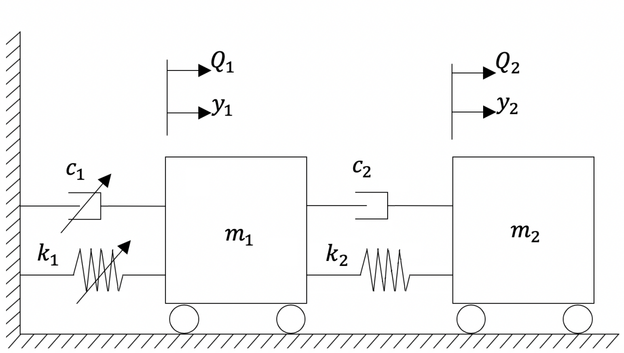


Fig. 6: Two degree-of-freedom nonlinear structural system subjected to non-stationary stochastic excitation

and

$$\mathbf{K}_{eq,(m_i,n_i),k} = 6(n-m)\Delta\omega \begin{bmatrix} \varepsilon_1 k_1 \sum_j S^{y_1 y_1}_{(m_j,n_j),k} & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{K}, \tag{80}$$

respectively. Further, the system parameter values $m_i = 1$, $c_i = 4.3$ and $k_i = 256$ ($i = 1, 2$), and the nonlinearity magnitude parameters $\varepsilon_1 = 2, \lambda_1 = 0.5$ are considered. It is also assumed for simplicity that $\mathbf{Q}_1(t) = \mathbf{Q}_2(t) = \mathbf{Q}(t)$ and the excitation parameter values are $S_0 = 10$ and $c_0 = 0.15$. Thus, solving the nonlinear set of equations formed by Eqs. (79-80) and Eqs. (33-34), leads to determining the evolutionary response power spectrum for the $2-$DOF system of Fig. 6. Figs. 7(a) and 7(b) represent the results for the generalized coordinates y_1 and y_2, also including the corresponding MIF.

Next, employing a redundant coordinates modeling for the formulation of the system governing equations of motion, the $2-$DOF system of Fig. 6 is decomposed in its partial subsystems, as shown in Fig. 8. The independent coordinates $\bar{x}_1, x_2$ and $\bar{x}_3$ describe the equations of motion of the two subsystems, which are also connected to each other by the constraint equation

$$x_2 = \bar{x}_1 + l_0 + d, \tag{81}$$

where l_0 is the unstreched length of the spring k_1 and d the physical length of mass m_1. Note, in passing, that the equations of motion for the system of Fig. 8 are formed in terms of relative coordinates, and thus, the connection between the independent coordinates $\bar{x}_1, \bar{x}_3$ and the generalized coordinates y_1, y_2 for the two different formulations are given by $\bar{x}_1 = y_1$ and $\bar{x}_3 = y_2$, respectively. Further, twice differentiating the system constraint equations, which are described by Eq. (81), matrix $\mathbf{A}$ of Eq. (25) becomes

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \tag{82}$$

whereas taking into account Eq. (26), $\mathbf{E} = \mathbf{L} = \mathbf{0}$ and $b = 0$.

Deriving the system governing equations of motion, the augmented mass, damping and stiffness matrices of Eqs. (21-23) take the form

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \; \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} 2.15 & 0 & 0 \\ 2.15 & 0 & 0 \\ 0 & 0 & 4.3 \\ 0 & 0 & 0 \end{bmatrix} \tag{83}$$

and

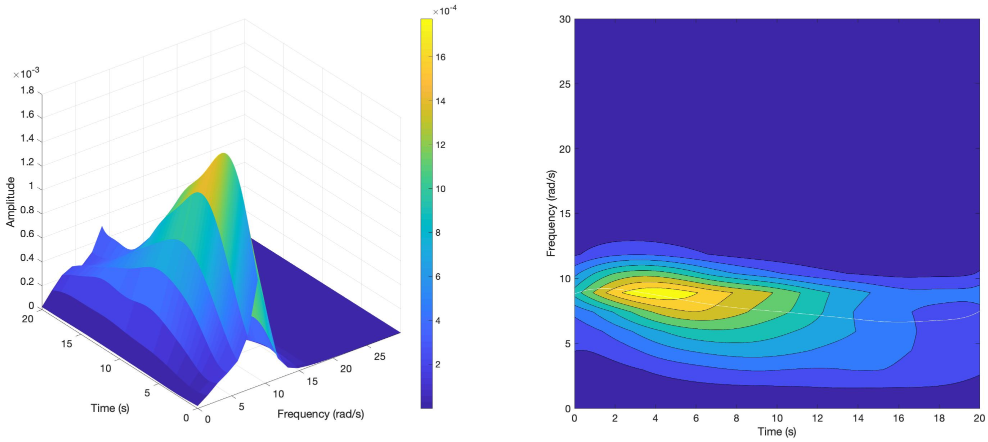
$$\bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 128 & 0 & 0 \\ 128 & 0 & 0 \\ 0 & 0 & 256 \\ 0 & 0 & 0 \end{bmatrix}, \tag{84}$$

respectively. Finally, the augmented system nonlinear vector defined in Eq. (43) is given by

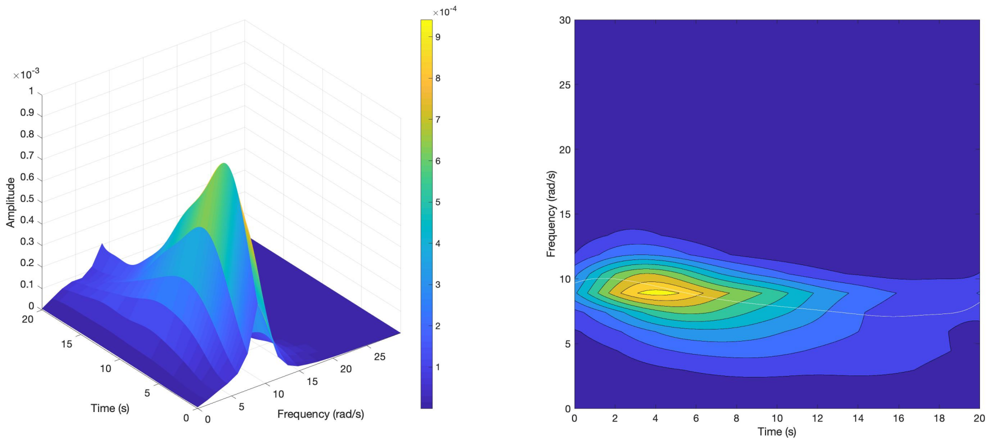
$$\bar{\mathbf{\Phi}}_{\mathbf{x}} = \begin{bmatrix} 0.5\varepsilon_1 k_1 \bar{x}_1^3 + 0.5\lambda_1 c_1 \dot{\bar{x}}_1^3 \\ 0.5\varepsilon_1 k_1 \bar{x}_1^3 + 0.5\lambda_1 c_1 \dot{\bar{x}}_1^3 \\ 0 \\ 0 \end{bmatrix}, \tag{85}$$

whereas the augmented excitation vector of Eq. (24) is given by

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_2 \\ 0 \end{bmatrix}. \tag{86}$$



(a)



(b)

Fig. 7: Response EPS of the nonlinear structural system of Fig. 6, subject to non-stationary stochastic excitation ($S_0 = 10, c_0 = 0.15$). (a) 1^{st} DOF y_1 ($\varepsilon_1 = 2, \lambda_1 = 0.5$); (b) 2^{nd} DOF y_2.

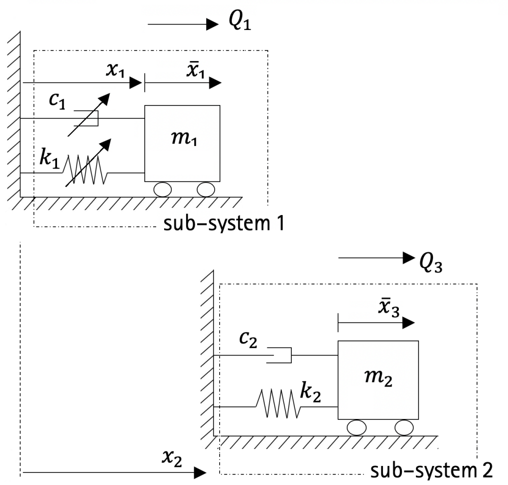


Fig. 8: Modeling the two degree-of-freedom system of Fig. 6 by using redundant coordinates.

Taking into account the system nonlinearity of Eq. (85), as well as the constraints expression of Eq. (81), matrices $\mathbf{D}_d$ and $\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$ in Eq. (51) become

$$\mathbf{D}_d = \begin{bmatrix} d_{1,(m,n),k} & d_{1,(m,n),k} & 0 & 0 \\ d_{1,(m,n),k} & d_{1,(m,n),k} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{87}$$

and

$$\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k} = \begin{bmatrix} s_{1,(m,n),k} & s_{1,(m,n),k} & 0 \\ s_{1,(m,n),k} & s_{1,(m,n),k} & 0 \\ 0 & 0 & s_{3,(m,n),k} \end{bmatrix}, \tag{88}$$

respectively, where

$$d_{1,(m,n),k} = \mathbb{E}[(\dot{x}_{1,(m,n),k})^4] + 3 \sum_{(i,j),i\neq m,j\neq n} \mathbb{E}[(\dot{x}_{1,(m,n),k})^2]\mathbb{E}[(\dot{x}_{1,(i,j),k})^2] \tag{89}$$

and

$$s_{\rho,(m,n),k} = \mathbb{E}[(\dot{x}_{i,(m,n),k})^2], \ \rho = 1, 3. \tag{90}$$

Further, the M-P inverse of matrix $\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k}$ in Eq. (88) is given by (47)

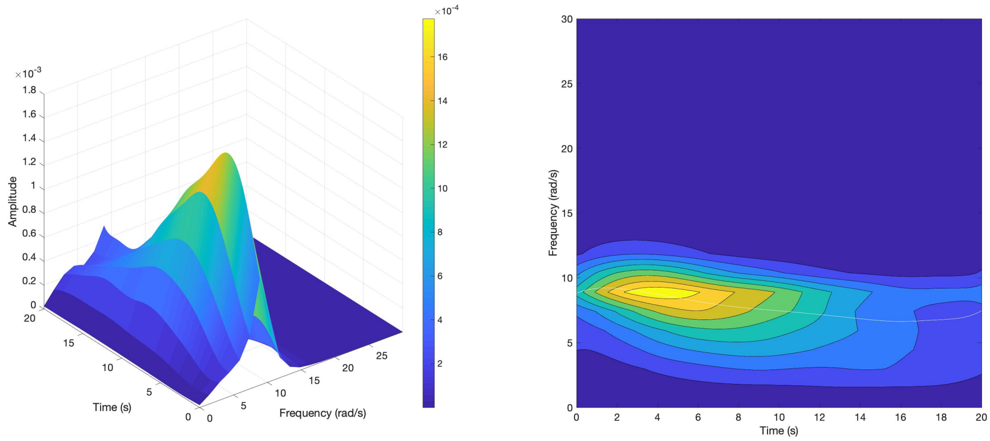
$$(\tilde{\mathbf{S}}^{\dot{\mathbf{x}}\dot{\mathbf{x}}}_{(m_i,n_i),k})^+ = \begin{bmatrix} 0.25s^{-1}_{1,(m,n),k} & 0.25s^{-1}_{1,(m,n),k} & 0 \\ 0.25s^{-1}_{1,(m,n),k} & 0.25s^{-1}_{1,(m,n),k} & 0 \\ 0 & 0 & s^{-1}_{3,(m,n),k} \end{bmatrix}. \tag{91}$$

Thus, substituting Eqs. (87) and (91) into Eq. (54), and manipulating, yields

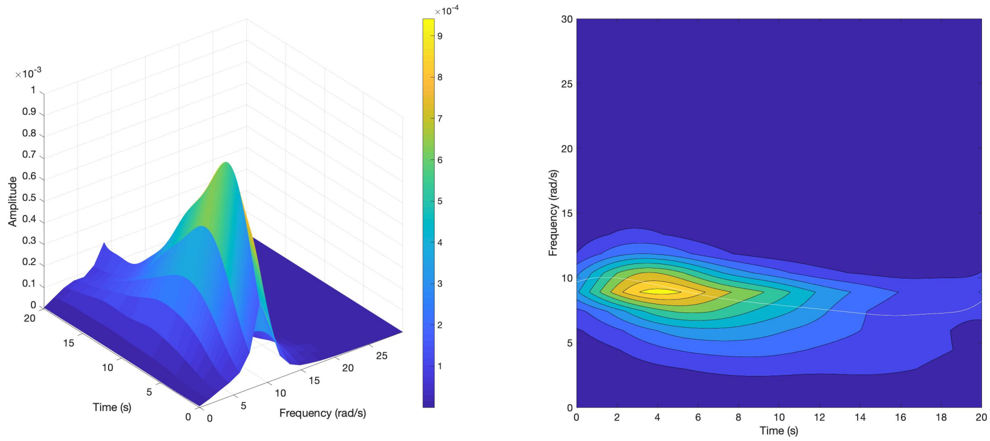
$$\bar{\mathbf{C}}_{eq,(m_i,n_i),k} = 6(n-m)\Delta\omega \begin{bmatrix} \frac{\lambda_1 c_1}{2}\sum_j S^{\dot{\bar{x}}_1\dot{\bar{x}}_1}_{(m_j,n_j),k} & \frac{\lambda_1 c_1}{2}\sum_j S^{\dot{\bar{x}}_1\dot{\bar{x}}_1}_{(m_j,n_j),k} & 0 \\ \frac{\lambda_1 c_1}{2}\sum_j S^{\dot{\bar{x}}_1\dot{\bar{x}}_1}_{(m_j,n_j),k} & \frac{\lambda_1 c_1}{2}\sum_j S^{\dot{\bar{x}}_1\dot{\bar{x}}_1}_{(m_j,n_j),k} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \bar{\mathbf{C}}_{\mathbf{x}}. \tag{92}$$

In a similar manner, the augmented equivalent linear stiffness elements are derived in closed form at each frequency band $(n-m)$ and time level k, as functions of the localized response EPS. They are given by

$$\bar{\mathbf{K}}_{eq,(m_i,n_i),k} = 6(n-m)\Delta\omega \begin{bmatrix} \frac{\varepsilon_1 k_1}{2}\sum_j S^{\bar{x}_1\bar{x}_1}_{(m_j,n_j),k} & \frac{\varepsilon_1 k_1}{2}\sum_j S^{\bar{x}_1\bar{x}_1}_{(m_j,n_j),k} & 0 \\ \frac{\varepsilon_1 k_1}{2}\sum_j S^{\bar{x}_1\bar{x}_1}_{(m_j,n_j),k} & \frac{\varepsilon_1 k_1}{2}\sum_j S^{\bar{x}_1\bar{x}_1}_{(m_j,n_j),k} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \bar{\mathbf{K}}_{\mathbf{x}}. \tag{93}$$



(a)



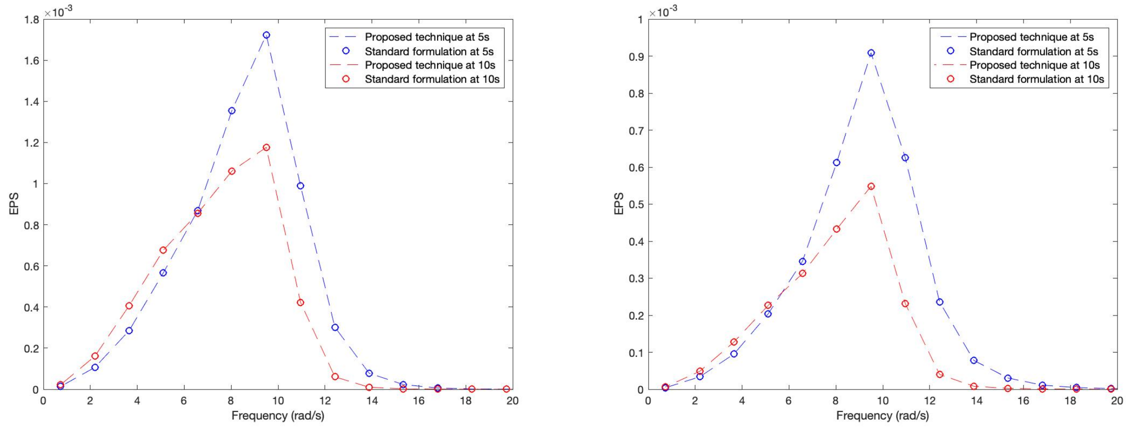
(b)

Fig. 9: Response EPS of the nonlinear structural system of Fig. 8 subject to non-stationary stochastic excitation ($S_0 = 10, c_0 = 0.15$). (a) 1^{st} DOF $\bar{x}_1$ ($\varepsilon_1 = 2, \lambda_1 = 0.5$); (b) 2^{nd} DOF $\bar{x}_3$.

Noticing that the matrix $\mathbf{R_x}$ of Eq. (32) has full rank, and thus, that the M-P HW-FRF of Eq. (40) holds, the estimation of the response EPS is attained by resorting to the recurcive linearization scheme which is applied to the nonlinear set of Eqs. (92-93) and Eqs. (40-41). The linearization process is initialized by considering the linear system response EPS, and continues until convergence. The scheme stops when the maximum values of the percentile difference of the equivalent elements over all frequency bands and time levels become smaller than 10^{-5}. In this regard, the estimated response EPS for the first DOF $\bar{x}_1$ is depicted in Fig. 9(a), whereas Fig. 9(b) shows the corresponding EPS for the independent to $\bar{x}_1$, coordinate $\bar{x}_3$. The MIF of Eq. (58) is also plotted in the figures for completeness. Comparing Figs. 9(a) and 9(b) with Figs. 7(a) and 7(b), respectively, it is seen that the corresponding results are in total agreement. This is also highlighted by Fig. 10, where the results obtained by applying the standard and alternative approaches are plotted for different time instants ($t = 5s$ and $t = 10s$). Thus, it is concluded that the herein developed M-P harmonic-wavelet-based statistical linearization scheme constitutes a generalization of the results in Ref. (13), to account for the case of systems with singular matrices subjected to non-stationary excitation.

5 Conclusion

In this paper, a generalized inverse matrix harmonic-wavelet-based technique has been developed for determining the response evolutionary power spectrum (EPS) of stochastically excited multi-degree-of-freedom (MDOF) linear and nonlinear structural systems with singular matrices. The singular matrices appear in the system governing equations of motion due to adopting a redundant coordinates modeling for their formulation. It can be argued, that



(a) (b)

Fig. 10: Response EPS of the nonlinear structural system of Figs. 6 and 8 at time instants $t = 5s$ and $t = 10s$, subject to non-stationary stochastic excitation ($S_0 = 10, c_0 = 0.15$). Comparison between standard formulation and the proposed technique. (a) 1^{st} DOF (y_1 vis-à-vis $\bar{x}_1$); (b) 2^{nd} DOF (y_2 vis-à-vis $\bar{x}_3$).

this approach relates to solutions of reduced computational cost, as well as enhanced modeling flexibility when the problem of forming the equations of motion of complex multibody systems is considered. However, singular matrices also hinder the application of standard solution treatments for estimating the system response EPS. In this regard, resorting to the Moore-Penrose (M-P) matrix inverse theory in conjunction with the generalized harmonic wavelets theory, a solution framework is developed herein for determining the evolutionary response spectra of such systems. Specifically, adopting the locally stationary wavelet representation of a stochastic process, and relying on the theory of the M-P generalized inverse of a singular matrix, an M-P localized in time and frequency, harmonic-wavelet-based frequency response function (M-P HW-FRF) has been constructed. Subsequently, employing the novel M-P HW-FRF, an input-output formula for determining the EPS of the system response has been derived. This can be construed as a generalization of a recently developed harmonic-wavelet-based input-output formula (13), to account for the case of systems with singular matrices. Further, for the case of nonlinear systems of this kind, a recently derived harmonic-wavelet-based statistical linearization technique (13) has also been generalized. First, an equivalent linear to the original nonlinear system has been defined. Subsequently, closed form solutions have been derived for the time and frequency dependent equivalent linear elements, and the equivalent linear system EPS has been estimated by resorting to the solution of a nonlinear set of equations. A linear and a nonlinear MDOF systems with singular matrices have been considered as numerical examples for assessing the validity of the developed methodology. The applicability of the proposed method to systems subject to a broad category of non-stationary excitations has been highlighted by considering excitations described by evolutionary non-separable spectra.

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Conflict of interests

The authors declare that they have no conflict of interest.

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